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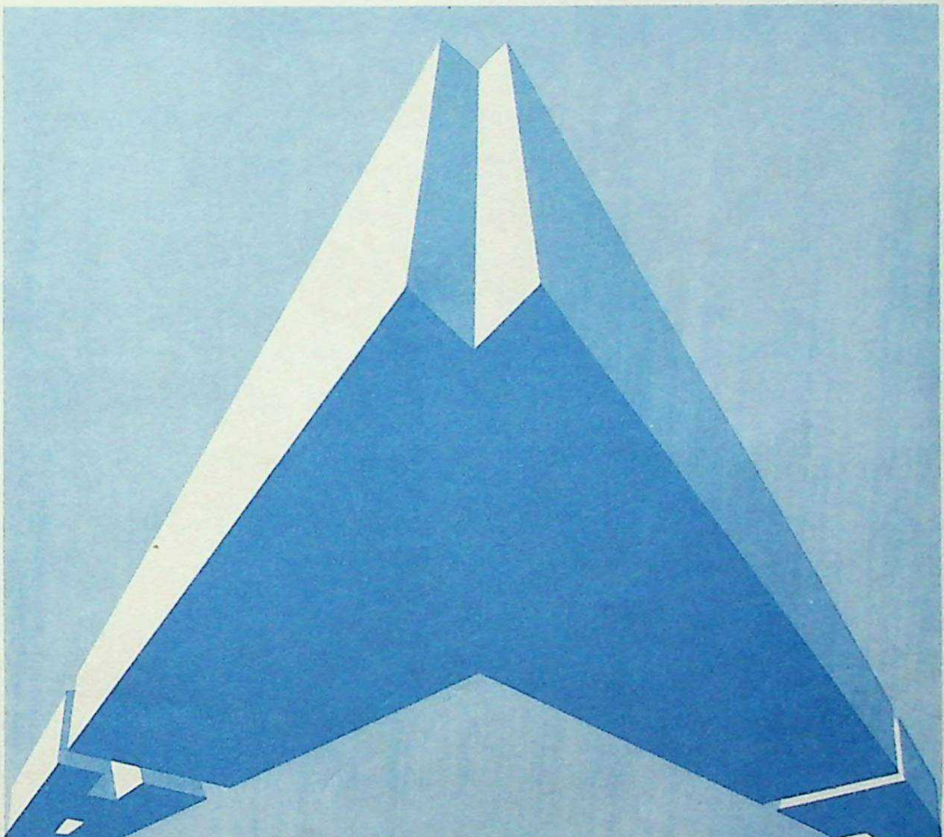

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ON CS-SEMIDEVELOPABLE SPACES

SUNG RYONG YOO

0. Instruction

In this paper cs-semidevelopable spaces are defined and shown to be the same as the semimetrizable spaces. Strongly cs-semidevelopable space are defined in a natural way and proved to coincide with an important class of semi-metric space, namely those in which "Cauchy sequence suffice". These space are shown to possess a few other interesting properties. Probably the most significant of these are that a space X is a cf-semistratifiable $w\Delta$-space if and only if it is cs-semidevelopable and that the image of a cs-semidevelopable space under a continuous pseudo open is cs-semidevelopable.

1. Cs-semidevelopable spaces

DEFINITION 1.1. (D1). A development for a space X is a sequeuce

$$\Delta=\{g_n|n\in N\}$$

of open covers of X such that $\{\mathrm{st}(x,g\ )|\ n\in N\}$ is a local base at x, for each $x\in X$. A space is developable if and only if there exists a development for the space,

DEFINITION 1.2. Let $\Delta=\{g_n|\ n\in N\}$ be a sequence of (not necessarily open) convers of space X,

(D2). Δ is a semidevelopment for X if and only if, for each $x\in X$, $\{st(x,g_n)n\in N\}$ is a local system of neighborhoods at x.

(D3). A semidevelopment of X is a strong-semidevelopment if and only if for each $M \subset X$ and $x \in \bar{M}$ there exists a descending sequence $\{G_n | n \in N\}$ such that $x \in G_n \in g_n$ and $G_n \cap M \neq \phi$.

(D4). A semidevelopment Δ for X is a point-finite semidevelopment if and only if for each $x \in X$ and for each positive integer n, x is contained in only a finite number of sets in g_n.

(D5). A semidevelopment Δ for X is a cs-semidevelopment if and only if for each convergent sequence $x_n \rightarrow x$ and for each open subset U containing $x \in X$, there is a positive integer k such that $x \in \mathrm{st}(x, g_k) \subset U$ and $\langle x_n \rangle$ is eventually in $\mathrm{st}(x, g_k)$.

A space is called semidevelopable if and only if there exists a semidevelopment for X. Similarly, X is called strongly (and/or point finite) semidevelopable if there exists a strong (and/or point-finite) semidevelopment for X.

Finally, a space X is called cs-semidevelopable if and only if there exists a cs-semidevelopment for X, Similarly that X is called strongly (and/or point-finite) cs-semidevelopable if and only if there exists a strong (and/or point-finite) cs-semidevelopment for X.

PROPOSITION 1.3. In order that a sequence $\Delta = \{g_n | n \in N\}$ of cover of a space X be a cs-semidevelopment it is necessary and sufficient that for each $M \subset X$ and $x \in \bar{M}$ there exists a sequence $\{G_n | n \in N\}$ such that

$$x \in G_n \in g_n \text{ and } G_n \cap M \neq \phi$$

PROOF. Straightforward from Definition 1.2.

For late use, we note that every (point-finite and/or strongly) cs-semidevelopable space has a (poin-finite and/or strong) cs-semidevelopment $\{g_n | n \in N\}$ having the property that $g_{n+1} < g_n$ for each positive integer $n \in N$. Hence, whenever the existence of a cs-semidevelopment is assumed in a theorem. We may assume that it has the property mentioned above cs-semidevelopments having this property shall be called refining cs-semidevelopments.

Definition 1.4. A metric on a space X is a function d:

$X \times X \to R$ (real numbers) satisfying the following conditions:

For each $x, y, z \in X$ and $\phi \neq M \subset X$,

(1) $d(x, x) = 0$

(2) $d(x, y) > 0$ if $x \neq y$

(3) $d(x, y) = d(y, x)$

(4) $d(x, z) \leq d(x, y) + d(y, z)$

(5) $x \in \bar{M}$ if and only if $d(x, M) = \inf\{d(x, m) | m \in M\} = 0$

Definition 1.5. A semi-metric on a space X is a function d: $X \times X \to R$ satisfying conditions (1), (2), (3) and (5) above. By a (semi-) metric space we mean a space X together with a specific (semi-) metric on X, In this paper, whenever the (semi-) metric is not specified it will be assumed to be denoted by the letter "d", the sphere about the point x of radius "ε" will be denoted by $S(x:\varepsilon)$. Note that spheres need not be open that $x \in \operatorname{Int} S(x:\varepsilon)$ if $\varepsilon > 0$.

It should be noted that in most of our theorem the T_0 property is assumed. This is usually done to insure that a cs-semidevelopable space satisfies (2) in the previous

definition which is satisfied in a semi-metric spaces.

DEFINITION 1.6. Let (X, d) be a semi-metric space. A sequence $\{x_n | n \in N\}$ in X is a Cauchy sequence if and only if for each $\varepsilon > 0$ there exists an integer N_0 such that $d(x_n, x_m) \langle \varepsilon$ whenever $m, n \rangle N_0$.

Note that because of the lack of the triangle inequality not all convergent sequences in a semi-metric space are necessarily Cauchy sequences.

2. Theorems for Cs-semidevelopable spaces

THEOREM 2.1. A space X is semi-metrizable if and only if it is a cs-semidevelopable space.

PROOF. Let $\Delta = \{g_n | n \in N\}$ be a refining cs-semidevelopment for the cs-semidevelopable space where, without loss of generality, $g_1 = \{X\}$. For $x, y \in X$, let $n(x, y)$ be the smallest integer n such that there is n$_0$ element of g_n containing both x and y. If n$_0$ such integer exists let $n(x, y) = \infty$

Define d: $X \times X \to R$ as follows. For $x, y \in X$, let $d(x, y) = 2^{-n(x, y)}$, where $2^{-\infty} = 0$. Then clearly, for every $x, y \in X$, $d(x, x) = 0$ and $d(x, y) = d(y, x)$, Also if $x \neq$y, then, since X satisfies(D5) in the previous Definition 1.2., there is an open set U containing one of the points, say x but not the other. Then there is an integer n such that $x \in st(x, g_n) \subset U$. Then $y \in U$ implies $y \in st(x, g_n)$ which implies $y \in st(x, g_i)$ for each $i \geq n$.

It follows that $n(x, y) \leq n$ and hence $d(x, y) \geq 2^{-n} > 0$.

New note that $S(x : 2^{-n}) = st(x, g_n)$ for each $x \in X$ and each integer n. For $y \in S(x : 2^{-n})$ if and only if $d(x, y) < 2^{-n}$ if and only if $n(x, y) > n$ if and only if there exists $G \in g_n$ such that $x, y \in G$ if and only if $y \in st(x, g_n)$. Now

let $M \subset X$. Then $X \in \bar{M}$ if and only if $st(x, g_n) \cap M \neq \phi$ for each integer n if and only if $S(x:2^{-n}) \cap M \neq \phi$ for each integer n if and only if $d(x, M)=0$ Hence, d is a semi-metric on X.

Conversely, assume that d is a semi-metric on X.

For each positive integer n, let g_n be the collection of all sets of diameter less than $1/n$. Then for each n, $S(x:1/n)=st(x, g_n)$. For let $y \in S(x:1/n)$. Then $G=\{x, y\} \in g_n$ implies $y \in st(x, g_n)$. On the other hand, let $y \in st(x, g_n)$. Then there is $G \in g_n$ such that $x, y \in G$, and therefore, $d(x, y) \leqq \text{diam } G < 1/n$ thus, $y \in S(x:1/n)$.

Now let U be an open set containing the point x. Then there is an integer n such that $x \in \text{Int } S(x:1/n) \subset S(x:1/n) \subset S(x_n:1/n) \subset U$. Therefore, $x \in \text{Int } st(x, g_n) \subset st(x, g_n) \subset st(x_n, g_n) \subset U$ and $\langle x_n \rangle$ is eventually in $st(x, g\ )$. Hence $\{g_n | n \in N\}$ is a cs-semidevelopment for X.

Corollary 2.2. Every cs-semidevelopable space is T_1.

Proof. Since every cs-semidevelopable space implies T_0 semi-developable and moreover T_0 semidevelopable spaces succeed T_1-space.

Theorem 2.3. In a cs-semidevelopable space the following conditions are equivalent:

(1) For each $M \subset X$ and each $x \in \bar{M}$, there exists a descending sequence of sets $\{G_n | n \in N\}$ of arbitrarily small diameters such that for each n, $x \in G_n$ and $x \in G_n \cap U \neq \phi$.

(2) For each $M \subset X$ and each $x \in \bar{M}$, there exists a Cauchy sequence in M converging to x.

(3) Every convergent sequence has a Cauchy Subsequence.

ence.

PROOF. Let d be a semi-metric on X since every cs-semidevelopable space implies a semi-metric space.

(1) implies(3). Let $S=\{x_n|n\in N\}$ be a sequence in X converging to the point $x\in X$. It $x_n=x$ for infinitely many n, then clearly we can define a Cauchy subsequence of S.

Otherwise let $M=\{x_n|n\in N\}\setminus\{x\}$. Then $x\in\bar{M}$ implies, by (1), that there is a descending sequence of sets $\{G_n|n\in N\}$ of arbitrarily small diameters such that for each n, $x\in G_n$ and $G_n\cap M\neq\phi$. We now define a subsequence of $\{x_n|n\in N\}$ inductively. Choose $x_{n_i}\in G_1\cap M$. Suppose x_{n_i} has been chosen for each $i=1,2,\ k-1$, such that $x_{n_i}\in G_i\cap M$ and $n_i>n_{i-1}$. Now observe that $G_k\cap M$ is infinite.

For suppose not: say $G_k\cap M=\{a_1,\cdots,\ a_m\}$. Then there exists $n_o>K$ such that diam $G_{n_o}<\min\{d(x,a_i)|i=1,2,\ m\}$ Clearly $a_i\notin G_{n_o}$ for each$=1,2,\cdots,\ m$. But then $M\cap G_{n_o}\subset M\cap G_k=\{a,\cdots a_m\}$
implies $M\subset G_{n_o}=\phi$, which is a contradiction.
Hence we can choose $x_{n_k}\in G_k\cap M$ such that $n_k>n_{k-1}$. Thus we have defined a subsequence $\{x_{n_k}|k\in N\}$ of S which is Cauchy. For let $\varepsilon>0$ be given. Then there is an integer N_0 such that diam $G_{N_o}<\varepsilon$. For $i,j\geq N_0$, We then have $x_{n_i}\in G_i\subset G_{N_o}$ and $x_{n_j}\ G_j\subset G_{N_o}$.

Thus $d(x_{n_i},x_{n_j})\leq$diam $G_{N_o}<\varepsilon$.

(3) implies(2): Assume $M\subset X$ and $x\in\bar{M}$. Since X is first countable there is a sequence $\{x_n|n\in N\}$ in M which converges to x.

By (3), this sequence has a Cauchy subsequence $\{x_{n_k}|k\in N\}$.

Then $\{x_{n_k}|k\in N\}$ is a Cauchy sequence in M converging

to x.

(2) implies (1): Let $M \subset X$ and assume $x \in \bar{M}$. Then, by (2), there is a Cauchy sequence $\{x_n | n \in N\}$ in M which converges to x. For each n, let $G_n = \{x_i | i \geq u\} \cup \{x\}$. Then $\{G_n | n \in N\}$ is a descending sequence of sets of arbitrarily small diameters such that for each n, $x \in G_n$ and $G_n \cap M \neq \phi$.

Definition 2.4. A space X is strongly semi-metrizable if and only if a semi-metric satisfying any one of the conditions of the previous theorem can be realized on X.

Such a semi-metric is called a strong semi-metric.

Theorem 2.5. A space X is strongly semi-metrizable if and only if it is a strongly cs-semidevelopable space.

Proof. Let d be a strong semi-metric for X then, by Theorem 2,3 d satis fying condition (1). Now consider the cs-semidevelopment defined in Theorem 2.1.

By the definition of Δ_d and the fact that d satisfies the condition (1), it follows immediately that Δ_d is a strong cs-semidevelopment, Conversely, let $\Delta = \{g_n | n \in N\}$ be a refining strong cs-semidevelopment for X. Let d_Δ be the semi-metric on X as defined in Theorem 2.1. Observe that with this semi-metric, diam $G \leqq 2^{-n}$ for each $G \in g_n$ and $n \in N$. Thus it follows the definition of a strong semi-development that d_Δ satisfies condition (1) of the previous theorem and hence all of the conditions.

Definition 2.6. A space X as a $w\Delta$-space if and only if there is a sequence $\{g_n | n \in N\}$ of open cover of X such that, for each $x \in X$, if $x_n \in st(x, g_n)$ for $n \in N$ then the

sequence $\langle x_n \rangle$ has a cluster point. Such a sequence of open covers is called a $w\Delta$-sequence for X.

THEOREM 2.7. A space X is a cf-semistratifiable $w\Delta$-space if and only if it is cs-semidevelopable.

PROOF. Let F be a cf-semistratification for a space X, and let $\Delta=\{g_n|n\in N\}$ is a $w\Delta$-sequence for the space X. We can take a $\mathrm{st}(x, g_n)$ such that $\mathrm{st}(x, g_n)\subset A_\alpha \subset F(k, U)$, Where A_α is an element of any filterbase in X.

Since from definition of filterbase, g_{n+1} is an open refinement of g_n for all n. Thus $\{\mathrm{st}(x, g_n)|n\in N\}$ is a local system of neighborhood at x, therefore $\{g_n|n\in N\}$ is a semidevelopment for X and moreover, there is a convergent sequence $\langle x_n \rangle$ is the space X since X is a $w\Delta$-space, there is a positive $k\in N$ such that $x\in st(x, g_n)$ and $x_n\in st(x, g_n)\subset U$, for all $n\in N$.

Hence the semidevelopable space implies a cs-semidevelopable space as desired.

Conversely, let $\{H_n|\ n\in N\}$ be an open covers of X, and let $\widehat{\mathscr{U}}=\{A_\alpha|\alpha\in\mathscr{A}\}$ be a convergent filter base for X. For each positive integer n, let $g_n=\{G|G=(\bigcap_{i=1}^{n} H_i)\cap(\bigcap_{i=1}^{n} A_{\alpha i})$, $H_i\in\widehat{H}_i$, $A_{\alpha i}\in\widehat{\mathscr{U}}\}$,

then $\{g_n|n\in N\}$ is a cs-semidevelopment for X. To show that $\{g_n|n\in N\}$ is a $w\Delta$-sequence with a cf-semistratification for X. We can choose a neighborhood $U(x)$ of x such that $x\in st(x, g_n)\subset U(x)$. Since $\{g_n|n\in N\}$ is a semidevelopment for X, and choose a sequence $\langle x_n \rangle$ such that for all n, $x_n\in st(x, g_n)$, then $x_n\in U(x)$ this implies that $\langle x_n \rangle$ converges to x since g_{n+1} is an open refinement of g_n for all $n\in N$. Hence there is $A_n\in g_n$ such that $x_n\in A_n$

$\subset st(x, g_n)$. Suppose the filter base $\mathscr{U}=\{A_\alpha|\alpha\in\mathscr{A}\}$ converging to x has a cluster point p such that $x\neq p$. Then clearly there is a positive integer k such that for a neighborhood V of p, $V(p)\cap st(x, g_k)=\phi$ Now for $n\geq k$, $A_\alpha \subset st(x, g_n)\subseteq st(x, g_k)$ for all $\alpha\geq\beta$, $\beta\in\mathscr{A}$ and so $A_\alpha\cap V(p)=\phi$ for all $\alpha\geq\beta$. This constracts the fact p is a clustor point of $\mathscr{U}$ Thus $\{g_n|n\in N\}$ is a cf-semistratifiable $w\Delta$-space.

Corollary 2.8. Let X be a reqular $w\Delta$-space. Then X is an α-space if and only if X is a cs-semidevelopable space

3. Mapping

Charles C. Alexander introduced the concept of pseudo map.

Definition 3.1. Let X and Y be topological spaces, Then a surjective map from X onto Y is pseudo-open if and only if for each $y\in Y$ and each open neighbor hood U of $f^{-1}(y)$ in X, $y\in\text{Int } f(U)$.

Theorem 3.2. The image of a cs-semidevelopable space under a continuous pseudo-open map is cs-semidelopable.

Proof. Let f be a continuous pseudo-open map from a cs-semidevelopable space X onto a space Y and $\Delta=\{g_n| n\in N\}$ a cs-semidevelopment for X. For each open V_n containing a point of Y and for all n, we can put

$$f^{-1}(V_n)=st(x, g_n).$$

Since Δ is a cs-semidevelopment for X and f is continuans, let U be any open set in X including $f^{-1}(V_n)$, then

there is an convergent sequence $\langle x_n \rangle$ converging a point x belonging to $f^{-1}(y)$ in $\widehat{U}$, where $\langle y_n \rangle$ converges to y in Y. On the other hand. by Definition **1.2.** there exists a $n_0 \in N$ such that $st(x, g_n)$ is contained in for all $n > n_0$ and $\langle x_n \rangle$ is eventually in $st(x, g_{n_0})$. That is, $y \in f(st(x, g_n)) \subset \mathrm{Int} f(st(x, g_{n_0}))$ and therefore g_n is contained in Int $f(st(x, g_{n_0}))$ for all $n > n_0$.

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ON THE NUMERICAL RANGES AND LUMER'S FORMULA

KYONG SOO KIM AND YOUNGOH YANG

1. Introduction

In [4], Kim and Yang defined a numerical range for the class of all numerically bounded (nonlinear) maps on a Hilbert C*-module, and gave some of the basic properties of such numerical range. In this paper which is a continuation of [4], we define the numerical range for the numerically bounded vector fields on the unit sphere of a Hilbert C*-module, and give additional properties of such numerical ranges. In particular we obtain an analogue of Lumer's formula for the class of Lipschitz maps.

Throughout this paper we let B be a unital C*-algebra, $B^{\#}$ its dual space, and X the Hilbert B-module with a B-valued inner product $\langle , \rangle$[5]. A Hilbert B-module X is assumed to have a vector space structure over the complex numbers **C** compatible with that of B in the sense that

$$\lambda(xb)=(\lambda x)b=x(\lambda b) \quad (x\varepsilon X,\ b\varepsilon B,\ \lambda\varepsilon \mathbf{C}).$$

We define the norm $\|\cdot\|_X$ on X by $\|x\|_X = \|\langle x, x\rangle\|^{\frac{1}{2}}$. We will use the following notations. $Q^*(X)$ is the vector space of all B^*-quasibounded maps. $W^*(X)$ is the vector space of all B*-numerically bounded maps. $L(X)$ is the Banach space of all bounded linear operators on X. we also denote the operator norm on $L(X)$ by $\|\cdot\|$.

and $|F|^*=1$ it follows that

$$\left\| \frac{x_n}{\|x_n\|_X} + \frac{F(x_n, f_n)}{\|x_n\|_X} \right\|_X \to 2. \tag{3}$$

But (3) and X uniformly convex imply

$$\frac{\|(\pi-F)(x_n, f_n)\|_X}{\|x_n\|_X} \to 0. \tag{4}$$

Hence from (4) we obtain

$$d^*(F) = \lim_{r\to\infty} \inf_{\Pi_r} \frac{\|(\pi-F)(x, f)\|_X}{\|x\|_X} = 0,$$

i.e., $1 \varepsilon \sum^*(F)$.

On $W^*(X)$, the following seminorm is defined:

$$\omega^*(F) = \lim_{r\to\infty} \sup_{\Pi_r} \frac{|f(<F(x, f), x>)|}{\|x\|_X^2 \|f\|} \text{ [4].}$$

PROPOSITION 2.3. The multivalued function $F \varepsilon W^*(X) \to \Omega^*(F)$ is upper semicontinuous, i.e., given an neighborhood V of $\Omega^*(F)$ there exists an $\varepsilon>0$ such that $\Omega^*(G) \subset V$ for $G \varepsilon W^*(X)$, $\omega^*(F-G)<\varepsilon$.

PROOF. Suppose $\omega^*(G_n-F) \leqq \frac{1}{n}$, $z_n \varepsilon \Omega^*(G_n)$, $z_n \to z$. We will show that $z \varepsilon \Omega^*(F)$. It can be easily seen that this property implies the upper semicontinuity of $\Omega^*(F)$. By the definition of the seminorm $\omega^*(\cdot)$ we find $c_n>0$ such that

$$|f(<G_n(x,f)-F(x,f), x>)| \leqq \left(\frac{2}{n}\right) \|x\|_X^2 \|f\|$$

for $(x, f) \varepsilon \Pi_0$, $\|x\|_X \geqq c_n$. By the definition of a B*-numerical range we find $(x_n, f_n) \varepsilon \Pi_0$, $\|x_n\|_X \geqq n+c_n$ such that

$$|f_n(<(z_n\pi-G_n)(x_n, f_n), x_n>)| \leqq \left(\frac{1}{n}\right) \|x_n\|_X^2 \|f_n\|.$$

Hence $|f_n(<(z\pi-F)(x_n, f_n), x_n>)| \leq |f_n(<(F-G_n)(x_n, f_n),$

$$x_n>)|$$
$$+|f_n(<(G_n-z_n\pi)(x_n,\ f_n),\ x_n>)|+|z_n-z|\,\|x_n\|_X^2\|f_n\|$$
$$\leqq(\frac{3}{n}+|z_n-z|)\,\|x_n\|_X^2\|f_n\|.$$

Letting $n\to\infty$ we see that $z\varepsilon\Omega^*(F)$.

As a consequence, the set $\{F\varepsilon W^*(X):\Omega^*(F)\ \neq\phi\}$ is closed in $W^*(X)$. Also the multivalued function $F\varepsilon Q^*(X)\to\Sigma^*(F)$ is upper semi-continuous.

We recall that a continuous map $P: X_0=X-\{0\}\to X$ is said to be B-numerically bounded, if the map $F:\Pi_0\to X$ given by $F(x,f)=P(x)$ is B*-numerically bounded. In this case the numbers $\omega^*(F)$, $\alpha^*(F)$ and the B*-numerical range $\Omega^*(F)$ are denoted by $\omega(P)$, $\alpha(P)$ and $\Omega(P)$ respectively[4]. We denoted by $W(X)$ the vector space of all B-numerically bounded maps on X_0.

Let $S=\{x\varepsilon X: \|x\|_X=1\}$ be the unit sphere in X, and let $\Phi: S\to X$ be a continuous map on S, i.e., a vector field on S. We say that Φ is B-numerically bounded, if the map $\tilde{\Phi}(x)=\|x\|_X\Phi(\|x\|_X^{-1}x)$, $x\neq 0$, is B-numerically bounded. In this case we let $\omega(\Phi)=\omega(\tilde{\Phi})$, $\alpha(\Phi)=\alpha(\tilde{\Phi})$ and $\Omega(\Phi)=\Omega(\tilde{\Phi})$.

If we set $\Pi=\{(x,\ f)\ \varepsilon\ X\times B^{\#}: \|x\|_X=\|f\|=f(<x,\ x>)=1\}$, then Π is a connected subset of $X\times B^{\#}$ with the norm$\times$weak* topology[6].

PROPOSITION 2.4. Let Φ be a B-numerically bounded vector field on S. Then

(a) $\omega(\Phi)\ =\ \sup_{\Pi}\ |g(<\Phi(u),\ u>)|$.

(b) $\alpha(\Phi)\ =\ \inf_{\Pi}\ |g(<\Phi(u),\ u>)|$.

(c) $\Omega(\Phi)=\{g(<\Phi(u),\ u>): (u,\ g)\ \varepsilon\ \Pi\}$.

2. Numerical range for nonlinear operators

If we set

$$\Pi_r = \{(x, f) \varepsilon X \times B^{\#} : \|x\|_X = \|f\| \geqq r,\ f(\langle x, x \rangle) = \|x\|_X^3\}$$
$$(r > 0)$$

and $\Pi_0 = \bigcup_{r>0} \Pi_r$, then each $\Pi_r (r>0)$ and Π_0 are connected subsets of $X \times B^{\#}$ with the norm$\times$weak* topology, unless X has dimension one over R[4]. From now on we shall assume that Π_0 has the norm$\times$weak* topology as a subset of $X \times B^{\#}$. Also we shall assume that X doesn't have dimension one over R.

PROPOSITION 2.1. Let $F: \Pi_0 \to X$ be a continuous map such that $\|F(x, f)\|_X = \|x\|_X$ for $(x, f) \varepsilon \Pi_0$. Then $z \varepsilon \Sigma^*(F)$ implies $|z| = 1$, where $\Sigma^*(F)$ denotes the B*-asymptotic spectrum of $F \varepsilon Q^*(X)$[4].

PROOF. Let $z \varepsilon \Sigma^*(F)$. Then by definition of $\Sigma^*(F)$ we can find $(x_n, f_n) \varepsilon \Pi_0$ such that $\|x_n\|_X \geqq n$ and

$$\|(z\pi - F)(x_n, f_n)\|_X \leqq \frac{1}{n} \|x_n\|_X,$$

where π denotes the natural projection of $X \times B^{\#}$ onto X.

Hence
$$\|F(x_n, f_n)\|_X - \frac{1}{n}\|x_n\|_X \leqq |z| \, \|x_n\|_X \leqq \|F(x_n, f_n)\|_X + \frac{1}{n} \|x_n\|_X.$$

Using the assumption on F,

$$(1 - \frac{1}{n}) \|x_n\|_X \leqq |z| \, \|x_n\|_X \leqq (1 + \frac{1}{n}) \|x_n\|_X.$$

Dividing by $\|x_n\|_X$ and letting $n \to \infty$ completes the proof.

We note that the B^*-numerical range $\Omega^*(F)$ of $F\varepsilon W^*(X)$ is a nonempty compact connected subset of **C**, and $\Sigma^*(F)\subseteqq\Omega^*(F)$ $(F\varepsilon Q^*(X))$ [4]. Also we recall that a Banach space $(Y,\|\cdot\|)$ is said to be uniformly convex, if whenever $x_n\ \varepsilon\ Y$, $y_n\ \varepsilon\ Y$, $\|x_n\|\leqq 1$, $\|y_n\|\leqq 1$ and $\|x_n+y_n\|\to 2$, then $\|x_n-y_n\|\to 0$.

PROPOSITION 2.2. If X is uniformly convex and $F\varepsilon Q^*(X)$, then $\{\lambda\varepsilon\Omega^*(F):|\lambda|=|F|^*\}\subseteqq\Sigma^*(F)$, where $|\cdot|^*$ denotes the seminorm on $Q^*(X)$[4].

PROOF. Let $\lambda\varepsilon\Omega^*(F)$and $|\lambda|=|F|^*$. We may assume that $\lambda\neq 0$, for otherwise $\tilde{F}=0\varepsilon\tilde{Q}^*(X)$, the normed space of all equivalence classes of B*-quasibounded maps, i.e.,

$$\tilde{Q}^*(X)=Q^*(X)/N(|\cdot|^*)\ [4]$$

and the result follows immediately. Since we may replace F by $\lambda^{-1}F$, there is no loss of generality in assuming that $|F|^*=\lambda=1$.

Now, there exists $(x_n,\ f_n)\ \varepsilon\ \Pi_n$ such that

$$\frac{f_n(<F(x_n,\ f_n),\ x_n>)}{\|x_n\|_X^2\|f_n\|}\to 1$$

as $n\to\infty$ and therefore

$$\frac{f_n(<(\pi+F)(x_n,\ f_n),\ x_n>)}{\|x_n\|_X^2\|f_n\|}\to 2. \qquad (1)$$

Since

$$1+\frac{\|F(x_n,\ f_n)\|_X}{\|x_n\|_X}\geqq\frac{\|(\pi+F)(x_n,\ f_n)\|_X}{\|x_n\|_X}$$

$$\geqq\frac{|f_n(<(\pi+F)(x_n,\ f_n),\ x_n>)|}{\|x_n\|_X^2\|f_n\|}, \quad (2)$$

PROOF. (a) and (b) follow from

$$\frac{f(<\tilde{\phi}(x),\ x>)}{\|x\|_X^2\|f\|}=\frac{f(<\|x\|_X\Phi(\|x\|_X^{-1}x),\ x>)}{\|x\|_X^2\|f\|}$$

$$=g(<\Phi(u),\ u>),$$

where $u=\|x\|_X^{-1}x$, $g=\|f\|^{-1}f$ and $(u,\ g)\ \varepsilon\ \Pi$. Now (c) becomes evident.

PROPOSITION 2.5. Let F be a continuous mapping of S into X, and let $W_B(F)=\{f(<Fx,\ x>):(x,\ f)\ \varepsilon\ \Pi\}$. Then $W_B(F)$ is connected.

PROOF. This follows from Corollary 3.4[6].

As a conseqence we see that $\Omega(\Phi)$ coincides with the closure $\overline{W_B(\Phi)}$ of the B-spatial numerical range $W_B(\Phi)$ of a continuous map $\Phi:S\rightarrow X$.

3. A nonlinear version of Lumer's formula

In [6] Yang proved the Lumer's formula

$$\sup \operatorname{Re} W_B(T)=\lim_{\alpha\to 0+}\frac{\|I+\alpha T\|-1}{\alpha}$$

for any bounded linear operator T on X, where $W_B(T)$ denotes the B-spatial numerical range of T.

Our aim in this section is to prove a nonlinear verson of Lumer's formula for the class of Lipschitz maps. But before we do this, we are going to state an elementary result which is a generalization of the well known properties of the logarithmic norm for bounded linear operators on a Banach space.

LEMMA 3.1 [2]. Let Y be a Banach space, and let $C(Y)$ be a vector space of continuous maps $f:Y_0=Y-\{0\}\rightarrow Y$ such that $I\varepsilon C(Y)$. Let δ be a semi-norm defined on $C(Y)$

such that $\delta(I)=1$. If for every $f\varepsilon C(Y)$ we define

$$\delta'(f) = \lim_{\rho\to 0+} \frac{\delta(I+\rho f)-1}{\rho} \qquad (*)$$

then the limit (*) exists and satisfies the properties:

(a) $|\delta'(f)|\leqq\delta(f)$.

(b) $\delta'(\mu f)=\mu\delta'(f)$, $\mu\geqq 0$.

(c) $\delta'(f+g)\leqq\delta'(f)+\delta'(g)$.

(d) $|\delta'(f)-\delta'(g)|\leqq\delta(f-g)$.

LEMMA 3.2. If $P\varepsilon W(X)$, then

$$\sup \operatorname{Re} \Omega(P)\leqq\omega'(P). \qquad (1)$$

PROOF. From the inequality

$$\operatorname{Re}\frac{f(<P(x),\ x>)}{\|x\|_X^2\|f\|}\leqq\frac{1}{\rho}\ \left\{\frac{|f(<x+\rho P(x),\ x>)|}{\|x\|_X^2\ f\|} -1\right\},\ \rho>0$$

and the obvious fact

$$\sup \operatorname{Re}\ \Omega(P)=\lim_{r\to\infty}\ \sup_{\prod_r}\ \operatorname{Re}\frac{f(<P(x),\ x>)}{\|x\|_X^2\|f\|},$$

we obtain

$$\sup \operatorname{Re}\ \Omega(P)\leqq\frac{\omega(I+\rho P)-1}{\rho},\ \rho>0. \qquad (2)$$

Now (1) follows, if in (2) we let $\rho\to 0^+$.

On the vector space $Q(X)$ of all quasibounded maps on X, the following seminorm is defined:

$$|P|=\lim_{\|x\|_X\to\infty}\ \sup\frac{\|T_x\|_X}{\|x\|_X}[3].$$

THEOREM 3.3. If $P:X\to X$ is a Lipschitz map, i.e., there exists $k>0$ such that

$$\|P(x)-P(y)\|_X \leqq k\|x-y\|_X, \quad x, y \varepsilon X, \tag{1}$$

then $\sup \operatorname{Re} \Omega(P)=\omega'(P)=|P|'$. (2)

PROOF. Since, clearly $\omega'(P) \leqq |P|'$, from the previous lemma we see that it suffices to show that

$$|P|' \leqq \sup \operatorname{Re} \Omega(P). \tag{3}$$

Let $\mu=\sup \operatorname{Re} \Omega(P)$ and $\mu_r=\sup \operatorname{Re} \overline{\phi_P(\Pi_r)}$ $(r>0)$, where ϕ_P is a continuous map given by

$$\Phi_P(x, f)=\frac{f(<P(x), x>)}{\|x\|_X^2\|f\|}, (x, f) \varepsilon \Pi_0.$$

We have for $(x, f) \varepsilon \Pi_r$ $(r>0)$

$$\frac{\|(I-\rho P)(x)\|_X}{\|x\|_X} \geqq \left|\frac{f(<(I-\rho P)(x), x>)}{\|x\|_X^2\|f\|}\right|$$

$$=\left|1-\rho \frac{f(<P(x), x>)}{\|x\|_X^2\|f\|}\right|$$

$$\geqq 1-\rho \operatorname{Re} \frac{f(<P(x), x>)}{\|x\|_X^2\|f\|}$$

$$\geqq 1-\rho \sup \operatorname{Re} \overline{\phi_P(\Pi_r)} \geqq 1-\rho\mu_r$$

and using the fact $\lim_{r\to\infty} \mu_r=\mu$, we obtain

$$\frac{\|(I-\rho P)(x)\|_X}{\|x\|_X} \geqq 1-\rho\mu_r>0, \quad \|x\|_X \geqq r, \tag{4}$$

for all $\rho>0$ sufficiently small.

If we apply (1) we obtain

$$\|x+\rho P(x)\|_X \geqq \|x\|_X-\rho\|P(x)\|_X$$

$$\geqq \|x\|_X-\rho(\|P(0)\|_X+k\|x\|_X)$$

$$\geqq (1-k\rho)\|x\|_X-\rho\|P(0)\|_X.$$

Thus, if we let $0<\rho<\frac{1}{k}$ we see from this last inequality that we can choose $\|x\|_X \geqq r$ large enough so that

$$\|x+\rho P(x)\|_X \geqq r.$$

Hence we can apply (4) with $x+\rho P(x)$ instead of x and obtain

$$\|(I-\rho P)(I+\rho P)(x)\|_X \geqq (1-\rho\mu_r)\|x+\rho P(x)\|_X,$$

and

$$\|(I+\rho P)(x)-\rho P(I+\rho P)(x)\|_X \geqq (1-\rho\mu_r)\|x+\rho P(x)\|_X. \quad (5)$$

From (1) we obtain

$$\begin{aligned}\|(I+\rho P)(x)-\rho P(I+\rho P)(x)\|_X &\leqq \|x\|_X+\rho\|P(x) \\ &\quad -P(I+\rho P)(x)\|_X \\ &\leqq \|x\|_X+\rho k\|x-(I+\rho P)(x\|_X) \\ &= \|x\|_X+\rho^2 k\|P(x)\|_X.\end{aligned}$$

Thus we have

$$\|(I+\rho P)(x)-\rho P(I+\rho P)(x)\|_X \leqq \|x\|_X+\rho^2 k\|P(x)\|_X. \quad (6)$$

From (5) and (6) we get

$$\|x\|_X+\rho^2 k\|P(x)\|_X \geqq (1-\rho\mu_r)\|x+\rho P(x)\|_X$$

and hence

$$1+\rho^2 k\frac{\|P(x)\|_X}{\|x\|_X} \geqq (1-\rho\mu_r)\frac{\|x+\rho P(x)\|_X}{\|x\|_X}. \quad (7)$$

If in (7) we take the lim sup as $r\to\infty$

we obtain $1+\rho^2 k|P| \geqq (1-\rho\mu)|I+\rho P|$,

and

$$\frac{|I+\rho P|-1}{\rho} \leqq \frac{\rho k|P|+\mu}{1-\rho\mu}. \quad (8)$$

If in (8) we let $\rho \to 0^+$, we obtain (3), and this completes the proof.

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ON SEMIGROUP RINGS(I)

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In this paper K will be a field, S-a semigroup ring of the semigroup S with coefficients in K will be denoted by $K[S]$ (*c.f.* Clifford[2]).

We shall prove that if $K[S]$ is right Artinian with a cancellative semigroup S, then S must be finite. Thus, we get a result related to the Woods' Theorem[5] for semigroup ring case.

We recall that a semigroup S will be called cancellative if $G=SS^{-1}$ be a group and for all $s,t\varepsilon S$, there exist $s_1,t_1\ \varepsilon\ S$ such that $st^{-1}=t_1s_1^{-1}$.

We begin with the following well-known result which is a kind of common denominator theorem.

LEMMA 1. Let S be a cancellative semigroup and $G=SS^{-1}$ be a group. If $s_1,s_2,\cdots, s_n\varepsilon S$, then there exist $r_1,r_2,\cdots, r_n\varepsilon S$ and $s\varepsilon S$ such that $s_i^{-1}=r_is^{-1}$ for $1\leq i\leq n$.

PROOF. The proof will be by induction on n. For $n=1$, take $s=s_1^2$ and $r_1=s_1$. Assuming the result to be ture for all positive integers less than n. By induction hypothesis, there exist $r_1^*,r_2^*,\cdots,r_{n-1}^*\ \varepsilon\ S$ and $s^*\ \varepsilon S$ such that

$$s_i^{-1}=r_i^*s^*$$

for $1\leq i\leq n-1$. Since S is cancellative, there exist r_n, $r\varepsilon S$ such that $s_nr_n=s^*\ r=s$ say. Hence $s_n^{-1}=r\ s^{-1}$. Set $r_i=r_i{}^*r$ for $1\leq i\leq n-1$ so that $s_i^{-1}=r_i^*s^{*-1}=(r_i^*r)(r^{-1}s^{*-1})$

$$=(\overset{*}{r_i}\, r)(s^* r)^{-1}=r_i s^{-1}$$

for $1 \leq i \leq n-1$. This completes the induction and concludes the proof.

Okninski[4] showed that if a semigroup ring $K[S]$ is left and right Artinian, then S is finite. In order to consider the above theorem to the semigroup ring case with one sided Artinian, we start with the following.

LEMMA 2. Let K be a field and S- cancellative semigroup. If I is a right ideal of group ring $K[G]$, then $I \cap K[S]$ is a right ideal of $K[S]$ and $(I \cap K[S])\ K[G]=I$.

PROOF. It is easy to see that $I \cap K[S]$ is a right ideal of $K[S]$. Also we get $(I \cap K[S])K[G] \subseteq I\ K[G] \subseteq I$ because of $I \cap K[S] \subseteq I$.

On the other hand, let $\alpha=a_1g_1+a_2g_2+\cdots+a_ng_n$ be an element of I. Since $G=SS^{-1}$, there exist $s_i, t_i \varepsilon S$ such that $g_i=s_it_i^{-1}$ for $1 \leq i \leq n$. Thus $\alpha=a_1s_1t_1^{-1}+a_2s_2t_2^{-1}+\cdots+a_ns_nt_n^{-1}$.

From lemma 1, we get $t_i^{-1}=s_i'\ t^{-1}$ for $1 \leq i \leq n$. Set $s_is_i'=\alpha_i \varepsilon S$. Then we have $\alpha=(a_1s_1s_1'+\cdots+a_ns_ns_n')t^{-1}=(a_1\alpha_1+\cdots+a_n\alpha_n)t^{-1}$ and $\alpha t=a_1\alpha_1+\cdots+a_n\alpha_n \varepsilon\ I$ because of $\alpha \varepsilon\ I$. Since $(a_1\alpha_1+\cdots+a_n\alpha_n)\ \varepsilon I \cap K[S]$, $\alpha=(a_1\alpha_1+\cdots+a_n\alpha_n)t^{-1}$ $\varepsilon(I \cap K[S])K[G]$. Thus $I \subseteq (I \cap K[S])K[G]$.

Now we are in a position to prove one of our main results.

THEEOREM 3. Let K be a field and S a cancellative semigroup. If semigroup ring $K[S]$ is right Artinian, then S is finite.

PROOF. Let $I_1 \supseteq I_2 \supseteq \cdots$ be a descending chain of right

ideals of $K[G]$. Then $I_1 \cap K[S] \supseteq I_2 \cap K[S] \supseteq \cdots$ is also descending chain of right ideals of $K[S]$. Since $K[S]$ satisfies descending chain condition on right ideals of $K[S]$, there is an integer N such that $I_k \cap K[S] = I_N$ for all $k \geq N$. Since $(I_k \cap K[S])K[G] = I_k$, $I_k = I_N$ for all $k \geq N$. It means that $K[G]$ is right Artinian. By Connell[3, Theorem 3.1], G is finite. Hence S is also finite.

The ring R is called right perfect if R is semilocal (*i.e.* $R/J(R)$ is Artinian) and the Jacobson radical $J(R)$ is right T-nilpotent, or equivalently, R satisfies the descending chain condition on principal left ideals [1, Theorem 28.4, p. 315].

S. M. Woods [4] has shown that the groupring $R[G]$ is right(left) perfect if and only if R is right(left) perfect and G is finite.

Now let us consider the perfect semigroup ring case.

THEOREM 4. Let S be commutative cancellative semigroup. If $K[S]$ is left perfect, then S is finite.

PROOF. By Woods' result, we must show that $K[G]$ is left perfect. Let $\alpha_1, \alpha_2, \alpha_3, \cdots$ be elements of $K[S]$ and $\alpha_1 K[G] \supseteq \alpha_2 K[G] \supseteq \alpha_3 K[G] \supseteq \cdots$ be a descending chain of principal right ideals of $K[G]$. By lemma 1, $\alpha_i = \beta_i s_i^{-1}$ for some $\beta_i \varepsilon K[S]$ and $s \varepsilon S$. Thus we get $\beta_1 s_1^{-1} K[G] \supseteq \beta_2 s_2^{-1} K[G] \supseteq \cdots$ and $\beta_1 K[G] \supseteq \beta_2 K[G] \supseteq \cdots$ because of $S \subset K[G]$. Since $\beta_2 = \beta_2 \cdot 1 \varepsilon \beta_2 K[G] \subseteq \beta_1 K[G]$, $\beta_2 = \beta_1 \gamma$ for some $\gamma \varepsilon K[G]$. Hence $\gamma = \gamma_0 t^{-1}$ for some $\gamma_0 \varepsilon K[S]$ and $t \varepsilon S$. Thus, we have $\beta_2 = \beta_1 \gamma_0 t^{-1}$ and $t\beta_2 = \beta_2 t = \beta_1 \gamma_0 \varepsilon \beta_1 K[S]$. Finally we get $t\beta_2 K[S] \subseteq \beta_1 K[S]$.

By the same reason $\beta_2 t K[S] \supseteq t_1 t \beta_3 K[S]$ for some $t_1 \varepsilon S$. Continuing in this fashion, we get the descending

chain $\beta_1 K[S] \supseteq \tau_2\beta_2 K[S] \supseteq \tau_3\beta_3 K[S] \supseteq \cdots$ of principal right ideals of $K[S]$. Since $K[S]$ is left perfect, there exists an integer N such that $\tau_k\beta_k K[S] = \tau_N\beta_N K[S]$ for all $k \geq N$. Since S is commutative, $\beta_k\tau_k K[S] = \beta_N\tau_N K[S]$ for all $k \geq N$. Thus $\beta_k\tau_k K[S]K[G] = \beta_N\tau_N K[S]K[G]$ for all $k \geq N$ and $\beta_k\tau_k K[G] = \beta_N\tau_N K[G]$ for all $k \geq N$. So for all $k \geq N$, $\beta_k K[G] = \beta_N K[G]$ and $\beta_k s_k^{-1} K[G] = \beta_N s_N^{-1} K[G]$, we get $\alpha_k K[G] = \alpha_N K[G]$ for all $k \geq N$. Therefore $K[G]$ satisfies the descending chain condition on the principal right ideals of $K[G]$. Hence $K[G]$ is left perfect and the proof is complete.

Corollary. If K is any field, then $K[x]$ is not perfect.

PROOF. We have known that $S = \{x^i \mid i = 0, 1, 2, \cdots\}$ is a cancellative semigroup and $K[S]$ is just the polynomial ring $K[x]$ of x over K. So by the theorem $K[x]$ never perfect since S is finite.

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ON THE NUMERICAL RANGE AND HERMITIAN OPERATORS

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I. Introduction.

In[7], W.L. Paschke investigated right modules over a C^*-algebra B which possess a B-valued inner product respecting module action. Under this inner product, Yang defined a numerical range of an operator on this module. In this paper, which is a continuation of [8], we give analogous results of our numerical ranges as those on Banach spaces, and study its relation to spectra and various growth conditions on the resolvent. Also we define Hermitian operators in terms of our numerical range and study some results on these.

Throughout this paper we let B be a unital C^*-algebra, B^* the dual space of B, X the Hilbert B-module with a B-valued inner product $\langle\ ,\ \rangle$[7], $S(X)$ the unit sphere of X, i.e., the set of all $x \varepsilon X$ such that $\|x\|_X = \|\langle x, x\rangle\|^{\frac{1}{2}} = 1$, π_1 the projection of $X \times B^*$ onto X, $B(X)$ the set of all bounded linear operators on X, and Π the subset of $X \times B^*$ defined by

$$\Pi = \{(x, f)\ \varepsilon\ S(X) \times S(B^*):\ f(\langle x, x\rangle) = 1\}.$$

The *B-spatial numerical range* $W_B(T)$ of an operator $T \varepsilon B(X)$ is the set $W_B(T) = \{f(\langle Tx, x\rangle):\ (x, f) \varepsilon \Pi\}$[8]. This generalizes the classical concept of numerical range on a Hilbert space. The *B-spatial numerical radius* of an operator T is the number $w_B(T) = \sup\{|\lambda|:\ \lambda \varepsilon\ W_B(T)\}$.

Also we denote the action of B on a right B-module X by $(x, b) \to xb (x \varepsilon X, b \varepsilon B)$. A Hilbert B-module X is assumed to have a vector space structure over the complex numbers **C** compatible with that of B in the sense that $\lambda(xb) = (\lambda x)b = x(\lambda b)$ $(x \varepsilon X,\ b \varepsilon B,\ \lambda \varepsilon \mathbf{C})$. The algebra of all bounded linear operators on X which possess bounded adjoints with respect to the B-valued inner product will be denoted by $A(X)$, and without risk of confusion, we denote the operator norm on $B(X)$ by $\| \cdot \|$.

II. Spatial numerical ranges

LEMMA 2.1. *Let P be a subset of Π such that $\pi_1(P)$ is dense in $S(X)$ and $T \varepsilon B(X)$. Then*

(i) $\sup\{Re\ f(\langle Tx, x\rangle) : (x, f) \varepsilon P\} = \inf\{\frac{1}{\alpha} (\| I + \alpha T \| - 1) : \alpha > 0\} = \lim_{\alpha \to 0+} \frac{1}{\alpha}(\| I + \alpha T \| - 1)$.

(ii) $\sup\{Re\ f(\langle Tx, x\rangle) : (x, f) \varepsilon P\} = \sup\{\frac{1}{\alpha} \log \| \exp(\alpha T) \| : \alpha > 0\} = \lim_{\alpha \to 0+} \frac{1}{\alpha} \log \| \exp(\alpha T) \|$.

(iii) $\overline{co}\{f(\langle Tx, x\rangle) : (x, f) \varepsilon P\} = V(\mathrm{B}(X), T)$, *where $\overline{co}$ E and $V(A, a)$ denote the closed convex hull of E and the numerical range of a of a unital normed algebra A respectively.*

PROOF. (i) and (iii) See [8].

(ii) The proof is similar to that of Theorem 3.4[2].

By Theorem 2.6[2], we have $S_P(T) \subset V(B(X), T) = \overline{co}\ W_B(T)$ where $S_P(T)$ denotes the spectrum of T. However, the following stronger statement holds.

THEOREM 2.2. *Let $T \varepsilon B(X)$ be any operator having its*

adjoint. Then $S_P(T)\subset\overline{W}_B(T)$; *In particular* $\partial S_P(T)\subset\overline{W}_B(T)$. ∂ *stands for "boundary of".*

Proof. For each $T\varepsilon B(X)$, we have $S_P(T)=\Pi(T)\cup\Gamma(T)$ where $\Pi(T)$, $\Gamma(T)$ denote the approximate point spectrum, the compression spectrum of T respectively. If $\lambda\varepsilon\Gamma(T)$, then the range of $T-\lambda I$ is not dense so since $R(\lambda I-T)^{\perp}=N(\bar{\lambda}I-T^*)$, the range of $T-\lambda I$ has a nonzero orthogonal complement. Hence $\bar{\lambda}$ is an eigenvalue of T^* so that $\bar{\lambda}\,\varepsilon W_B(T^*)$ and therefore $\lambda\varepsilon W_B(T)$. On the other hand if $\lambda\varepsilon\Pi(T)$, then there exist unit vectors x_n such that $(T-\lambda I)x_n\to 0$. By the Hahn-Banach Theorem, there exists f_n in B* of unit norm so that

$f_n(\langle x_n, x_n\rangle)=1$. Then $|f_n(\langle Tx_n, x_n\rangle)-\lambda|=|f_n(\langle Tx_n-\lambda x_n, x_n\rangle)|\leqq\|Tx_n-\lambda x_n\|_X$. Thus $f_n(\langle Tx_n, x_n\rangle)\to\lambda$ as $n\to\infty$, and hence $\lambda\varepsilon\overline{W}_B(T)$. It is well known that $\partial S_P(T)\subset\Pi(T)$[6], thus in particular $\partial S_P(T)\subset\overline{W}_B(T)$.

Remark 2.3. $\|R_\lambda\|=\|(T-\lambda I)^{-1}\|\leqq d(\lambda,\overline{W}_B(T))^{-1}$ for $\lambda\notin\overline{W}_B(T)$ $(T\varepsilon A(X))$. For given $x\varepsilon X$ with $\|x\|_X=1$, there is an f in B^* such that

$\|f\|=f(\langle x, x\rangle)=1$, and then $\|(T-\lambda I)x\|_X\geqq|f(\langle(T-\lambda I)x, x\rangle)|=|f(\langle Tx, x\rangle)-\lambda|\geqq d(\lambda,\overline{W}_B(T))$. Hence $\|(T-\lambda I)x\|_X\geqq d(\lambda,\overline{W}_B(T))\|x\|_X$ and so $\|R_\lambda\|=\|(T-\lambda I)^{-1}\|\leqq d(\lambda,\overline{W}_B(T))^{-1}$ for $\lambda\notin\overline{W}_B(T)$.

Theorem 2.4. *If* $T\varepsilon A(X)$ *and* K *is a closed convex subset of the plane, then* $K\supset W_B(T)$ *if and only if* $\|(T-\lambda I)^{-1}\|\leqq d(\lambda, K)^{-1}$ *for* $\lambda\notin K$.

Proof. If $K\supset W_B(T)$, then Remark implies that

$\|(T-\lambda I)^{-1}\|\leqq d(\lambda,\overline{W}_B(T))^{-1}\leqq d(\lambda, K)^{-1}$ for $\lambda\notin K$.

Conversely, suppose that the resolvent of T satisfies the

indicated growth condition. To show that $W_B(T)\subset K$, it suffices to show that every half-plane H which contains K also contains $W_B(T)$. By a preliminary translation and rotation, we may suppose that H is the right half-plane, $Re\ z\geqq 0$. Since $H\supset K$, $\|(I+tT)^{-1}\|=t^{-1}\|(t^{-1}I+T)^{-1}\|\leqq 1$ for all $t>0$. Hence if $(x,f)\ \varepsilon\ \Pi$, then $Re\ f(\langle (I+tT)^{-1}x, x\rangle)\leqq \|f\|\|(I+tT)^{-1}\|\ \|x\|^2{}_X\leqq 1=f(\langle x,x\rangle)$, and thus $0\leqq Re\ f(\langle (I-(I+tT)^{-1})x,x\rangle)=Re\ f(\langle tT(I+tT)^{-1}x,x\rangle)$. Dividing by t and letting $t\to 0$ yields $Re\ f(\langle Tx,x\rangle)\geqq 0$. Since (x,f) is arbitrary, this shows that $W_B(T)\subset H$.

THEOREM 2.5. *Let* $S, T\ \varepsilon\ A(X)$, $0\notin \overline{W}_B(T)$ *and*

$E=\{\lambda\mu^{-1}:\ \lambda\varepsilon\overline{W}_B(S),\ \mu\varepsilon\overline{W}_B(T)\}$. *Then* $S_P(T^{-1}S)\subset E$.

PROOF. Let z be a complex number not belonging to E. Then there exists $d>0$ such that $|z\mu-\lambda|\geqq d(\lambda\varepsilon\overline{W}_B(S), \mu\varepsilon\overline{W}_B(T))$. Given $(x, f)\varepsilon\Pi$, we have $\|(zT-S)x\|_X\geqq|f(\langle(zT-S)x,x\rangle)|=|zf(\langle Tx,x\rangle)-f(\langle Sx,x\rangle)|\geqq d$ since $f(\langle Tx,x\rangle)\ \varepsilon\ W_B(T)$ and $f(\langle Sx,x\rangle)\ \varepsilon\ W_B(S)$. Similarly $\|(zT-S)^*x\|_X\geqq d$. By [2], we conclude that $zT-S$ is invertible. Since $0\notin\overline{W}_B(T)$ and $S_P(T)\subset\overline{W}_B(T)$, T is invertible. Therefore $zI-T^{-1}S$ is invertible, i.e., $z\notin S_P(T^{-1}S)$.

The B-numerical index of X is the real number $n_B(X)$ defined by $n_B(X)=\inf\{\mathscr{W}_B(T):\ T\ \varepsilon\ B(X),\ \|T\|=1\}$. It is obvious that $\frac{1}{e}\leqq n_B(X)\leqq 1$ by Theorem 4.1[2]. It has long been known that for a complex Hilbert space X of dimension greater than one, $n_B(X)=\frac{1}{2}$[6].

Given $x\ \varepsilon\ S(X)$, let $D(B,\langle x,x\rangle)=\{f\ \varepsilon\ S(B^*):\ f(\langle x,x\rangle)=1\}$ and $W_B(T,x)=\{f(\langle Tx,x\rangle):\ f\ \varepsilon\ D(B,\langle x,x\rangle)\}$ $(T\ \varepsilon\ B(X))$. Then we see that $W_B(T)=\cup\{W_B(T,x):\ x\ \varepsilon\ S(X)\}$.

An application of the Hahn-Banach Theorem shows that for each $x \varepsilon S(X)$, we have $D(B, \langle x, x \rangle) \neq \phi$. In the weak* topology, $D(B, \langle x, x \rangle)$ is a closed convex subset of the unit ball in B^* and hence compact. Since $D(B, \langle x, x \rangle)$ is convex, $D(B, \langle x, x \rangle)$ is connected in any topology which makes B^* a topological linear space because in any such topology, $t \to tf + (1-t)g, 0 \leqq t \leqq 1$ is a continuous function. Using Lemma 15.7[3] and the fact that the sets $W_B(T, x)$ are nonvoid compact convex subsets of **C**, we obtain the following results;

LEMMA 2.6. *The mapping* $x \to W_B(T, x)$ *is an upper semicontinuous mapping of* $S(X)$ *with the norm topology into the nonvoid compact convex subset of* **C**.

From this fact, we can prove that $W_B(T)$ is connected for each $T \varepsilon B(X)$, unless X has dimension one over R.

In terms of the Hausdorff metric for compact subsets of the plane, we obtain the continuity of the function $\overline{W}_B$.

THEOREM 2.7. *The function* $\overline{W}_B$ *is continuous with respect to the uniform operator topology.*

PROOF. If $\|S-T\| < \varepsilon$ and $(x, f) \varepsilon \Pi$, then $|f(\langle (S-T)x, x \rangle)| \leqq \|S-T\| < \varepsilon$, and therefore $f(\langle Sx, x \rangle) = f(\langle Tx, x \rangle) + f(\langle (S-T)\ x,\ x \rangle) \ \varepsilon\ W_B(\mathrm{T}) + (\varepsilon)$. It follows that $\overline{W}_B(S) \subset \overline{W}_B(T) + (\varepsilon)$. Similarly $\overline{W}_B(T) \subset \overline{W}_B(\mathrm{S}) + (\varepsilon)$.

Given $T_1, \cdots, T_n \ \varepsilon\ B(X)$, we define *the B-joint numerical range of* $T = (T_1, \cdots, T_n)$ by $W(T) = \{(f(\langle T_1 x, x \rangle), \cdots, f(\langle T_n x, x \rangle)) : (x, f) \varepsilon \Pi\}$. Clearly $W_B(T)$ is a bounded subset of $\mathbf{C}^n$. We say that $z = (z_1, \cdots, z_n)$ is in *the joint point spectrum* of T if there exists some nonzero element $x \varepsilon X$ such that $T_i x = z_i x (i = 1, \cdots, n)$[2]. It is obvious that the B-joint numerical range of a n-tuple T of oper-

ators includes the joint point spectrum of T.

In this paper, we have the following two problems;

(1) For what operators $T \varepsilon B(X)$ is $W_B(T)$ a closed set?

(2) Characterize those Hilbert B-modules X such that $W_B(T)$ is convex for every $T \varepsilon B(X)$.

Ⅲ. Hermitian operators

DEFINITION 3.1. An operator $T \varepsilon B(X)$ is said to be B-Hermitian if $W_B(T)$ is real. We denote by $H(X)$ the set of B-Hermitian operators of $B(X)$.

It is obvious from the definition that $H(X)$ is a real Banach space, and $i(ST-TS) \varepsilon H(X)$ if T, $S \varepsilon H(X)$. Furthermore since $\mathscr{W}_B(\cdot)$ is equivalent to $\|\cdot\|$, any operator $T \varepsilon H(X)$ for which T and iT are both B-Hermitian must be equal to zero.

THEOREM 3.2. *Let T be an operator in $B(X)$. Then the following statements are equivalent*:

(i) T is B-Hermitian.

(ii) $I+i\alpha T\|=1+0(\alpha)$ an $\alpha\to 0$, with α real.

(iii) $\exp(i\alpha T)=1$ for real α.

(iv) $\exp(i\alpha T)\|\leqq 1$ for real α.

PROOF. Since $W(\alpha+\beta T)=\alpha+\beta W_B(T)$, T is B-Hermitian if and only if $\sup Re\ W_B(iT)=\sup\ Re\ W_B(-iT)=0$. Hence the equivalence of (i) and (ii) follows from Lemma 2.1 (i). Also (iii) implies (i) by Lemma 2.1 (ii). If (i) holds, then again by Lemma 2.1(ii), $0=\sup\{\frac{1}{|\alpha|}\log\|\exp(i\alpha T)\|:\alpha\neq 0\}$. (*) Now $1=\|I\|\leq\|\exp(i\alpha \mathrm{T})\|\ \|\exp(-i\alpha \mathrm{T})\|$ $(\alpha \varepsilon \mathrm{R})$. If, for some real β, we have $\|\exp(i\beta T)\|\neq 1$, the last inequality shows that there is a real number $\gamma\neq 0$ such

that $\|\exp(i\ \gamma\ T)\|>1$, contradicting(*). The argument just given also shows that (iii) and (iv) are equivalent. This completes the proof.

With the norm induced from $B(X)$, $A(X)$ is a C^*-algebra. It is easy to show that $T \varepsilon A(X)$ is B-Hermitian if and only if $T=T^*$.

PROPOSITION 3.3. *If $T \varepsilon B(X)$ can be written in the from $T=R+iJ$ with R and J B-Hermitian operators in $H(X)$, then R and J are uniquely determined.*

PROOF. If R' and J' are B-Hermitian operators in $B(X)$ and $T=R'+iJ'$, then $W_B(R-R')=iW_B(J'-J)=\{0\}$. Thus $R=R'$ and $J=J'$.

We let $J_B(X)=\{T+iS\colon T, S \varepsilon H(X)\}$, and we may define a mapping $*$ from $J_B(X)$ to itself by $(T+iS)^*=T-iS(T, S\varepsilon H(X))$. It is easy to show that $J_B(X)$ with the norm of $B(X)$ is a Banach space and $*$ is a continuous linear involution on $J_B(X)$.

DEFINITION 3.4. An operator $T \varepsilon B(X)$ is said to be B-positive if $W_B(T)\subset R^+$. We denote by $P(X)$ the set of all B-positive operators of $B(X)$.

In the real Banach space $H(X)$, the set $P(X)$ is a proper closed cone in which I is an interior point.

PROPOSITION 3.5. *Let $T \varepsilon P(X)$ and $0 \notin W_B(T)$. Then T is an interior point of $P(X)$ in $H(X)$.*

PROOF. Set $\lambda=\inf\{\alpha\colon \alpha \varepsilon W_B(T)\}$. Then $\lambda>0$ and $W_B(T+S)\subset W_B(T)+W_B(S)\subset W_B(T)+[-\|S\|, \|S\|]$ for any $S \varepsilon H(X)$. Thus if $\|S\|<\lambda$, we have $W_B(T+S)\subset R^+$, and the proposition follows.

We shall call pencil of elements of an algebra A the

set of linear combinations a-λb where $a \varepsilon A$, $b \varepsilon A$ and λ is a scalar parameter.

DEFINITION 3.6. Let T and S be elements of $B(X)$. The set $W_B^S(T)=\{\lambda: (E(x, f) \varepsilon \Pi) (f(\langle(T-\lambda S)x, x\rangle)=0)\}$ is called the *B-spatial numerical range of the pencil* $T-\lambda S$.

Under the additional assumption that $S \varepsilon H(X)$ with $S>0$, it is readily seen that

$$W_B^S(T)=\left\{\frac{f(\langle Tx, x\rangle)}{f(\langle Sx, x\rangle)} : (x, f) \varepsilon \Pi\right\}.$$

Definition 3.6 generalizes the B-spatial numerical range of an operator T since $W_B(T)=W_B^I(T)$. Generalizing B-spatial numerical radius, we also introduce

$\mathscr{W}_B^S(T)=\sup\{|\lambda|: \lambda \varepsilon W_B^S(T)\}$ and we have, in particular, $\mathscr{W}_B(T)=\mathscr{W}_B^I(T)$.

PROPOSITION 3.7. *Let $S \varepsilon H(X)$ and $S>0$. Then*

$$\mathscr{W}_B^S(T)= \inf_{\lambda>0} \{-\lambda S<T<\lambda S\} \quad (T \varepsilon B(X)).$$

PROOF. Obvious.

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EVOLUTION EQUATION OF TYPE $\frac{du}{dt}+A\beta u \ni 0$ IN $L^\infty(\Omega)$

KI SIK HA

1. Introduction.

Let Ω be a measure space of bounded measure. For $1\leq p\leq+\infty$, $L^p(\Omega)$ denotes the Lebesgue space of Ω with norm $\|\cdot\|_p$. Let A be an operator of $L^p(\Omega)$ and $\beta:\Omega\times R\to\mathscr{P}(R)$ a mapping. Define the operator $A\beta$ of $L^p(\Omega)$ by

$$A\beta=\{[u,w]\varepsilon L^p(\Omega)\times L^p(\Omega)|\text{there exists } v\varepsilon L^p(\Omega) \text{ such that } [v,w]\varepsilon A \text{ and } a.e.\ x\varepsilon\Omega,\ v(x)\varepsilon\beta(x,u(x))\}.$$

The purpose of this paper is to study the Cauchy problem

$$\frac{du}{dt}+A\beta u \ni 0,\ u(0)=u_0$$

in $L^\infty(\Omega)$.

Suppose that the following conditions are satisfied:

(H1) A is an m-accretive operator of $L^\infty(\Omega)$ and monotone in $L^2(\Omega)$,

(H2) $a.e.\ x\varepsilon\Omega,\ r\varepsilon R\to\beta(x,r)\varepsilon\mathscr{P}(R)$ is maximal monotone in R,

(H3) for every $r\varepsilon R$, there exists $v\varepsilon L^\infty(\Omega)$ such that $a.e.$ $x\varepsilon\Omega,\ v(x)\varepsilon\ \beta(x,r)$,

(H4) for every $f\varepsilon L^\infty(\Omega)$ and $\lambda>0$, there exists at most one solution $u\varepsilon L^\infty(\Omega)$ of $u+\lambda A\beta u\in f$.

THEOREM 1. ([5])
Let (H1)-(H4) be satisfied. Suppose $A\beta\neq\phi$. Then $A\beta$ is m-accretive in $L^\infty(\Omega)$.

2. Solutions of $\frac{du}{dt}+A\beta u \ni 0$ in $L^\infty(\Omega)$.

By [1] and Theorem 1, we have the following theorem on the integral solution in the sense of Bénilan of the equation:

THEOREM 2. Let (H1)-(H4) be satisfied. Suppose $A\beta \neq \Phi$. Let $g\varepsilon L^1(0,T;L^\infty(\Omega))$ for $T>0$. Then for every $u_0 \varepsilon\overline{D(A\beta)}$, there exists a unique integral solution u on $[0, T]$ of $du/dt+A\beta u \ni g$, $u(0)=u_0$. Moreover $u(t)\ \varepsilon\overline{D(A\beta)}$ for every $t\varepsilon[0,T]$.

Suppose the following condition is satisfied:

(HC) A is m-accretive in $L^\infty(\Omega)$ and cyclically monotone in $L^2(\Omega)$ ([2]).

REMARK 3. ([4]).

Let (HC) be satisfied. Then (H1) is satisfied and there exists a proper lower semi-continuous convex function ϕ defined on $L^2(\Omega)$ into $(-\infty,+\infty]$ with $\partial\phi=A_2$, where A_2 is the closure of A in $L^2(\Omega)$.

THEOREM 4.

Let (HC), (H2)–(H4) be satisfied. Then for every $u_0\varepsilon D(A\beta)$, there exists $u\varepsilon\mathscr{C}([0,T];L^\infty(\Omega))$ such that

$$\begin{cases} \dfrac{du}{dt}(t)\varepsilon L^\infty(0,\mathrm{T};\ \sigma(L^\infty(\Omega),L^1((\Omega)))) \\ \dfrac{du}{dt}+A\beta u(t)\ni 0 \text{ a.e. on } (0,T) \\ u(0)=u_0 \end{cases}$$

for $T>0$.

PROOF. By Theorem 1, $A\beta$ is m-accretive in $L^\infty(\Omega)$. Consider the approximation equation

$$(1)\qquad \begin{cases} \dfrac{u(t)-u_\varepsilon(t-\varepsilon)}{\varepsilon}+A\beta u_\varepsilon(t)\ni 0 & (t>0) \\ u(t)=u_0 & (t\leq 0) \end{cases}$$

for every $\varepsilon>0$ in $L^\infty(\Omega)$. The equation (1) is written by

$$\begin{cases} u_\varepsilon(t)=(I+\varepsilon A\beta)^{-1}u_\varepsilon(t-\varepsilon) & (t>0) \\ u_\varepsilon(t)=u_0 & (t\leq 0) \end{cases}$$

and it has at most one solution

$$u_\varepsilon(t)=(I+\varepsilon A\beta)^{-1}u_0 \text{ for every } t\varepsilon((k-1)\varepsilon, k\varepsilon]$$

or

$$u_\varepsilon(t)=(I+\varepsilon A\beta)^{-[t/\varepsilon]}u_0 \text{ for every } t>0.$$

By [3], $u_\varepsilon(t)$ converges to $u(t)$ in $L^\infty(\Omega)$ uniformly on every compact interval of $[0,\infty)$ as $\varepsilon\to 0+$ and if we put $u(t)=S(t)u_0$, then $S(t)$ is a nonlinear semigroup of contraction generated by $A\beta$. By [1], the limit $u(t)$ is the integral solution on $(0,\infty)$ of $\dfrac{du}{dt}+A\beta u\in 0$, $u(0)=u_0$ and $\|u(t)-u(s)\|_\infty\leq|t-s|\,\|A\beta u_0\|_\infty$ for every $t,s\geq 0$. Thus *a.e.* $t\varepsilon(0,\infty)$, $u(t)$ is weakly differentiable in $L^\infty(\Omega)$. We put

$$w_\varepsilon(t)=\frac{u_\varepsilon(t)-u_\varepsilon(t-\varepsilon)}{\varepsilon}.$$

For every $t\geq 0$,

$$(2)\qquad \|w_\varepsilon(t)\|_\infty=\frac{1}{\varepsilon}\|(I+\varepsilon A\beta)^{-[t/\varepsilon]}u_0-(I+\varepsilon A\beta)^{-[t/\varepsilon]+1}u_0\|_\infty$$
$$\leq\|A\beta u_0\|_\infty.$$

Hence w_ε is bounded in $L^\infty((0,\infty)\times\Omega)$. Let $v_\varepsilon(t)\varepsilon L^\infty(\Omega)$ such that

$$(3)\qquad w_\varepsilon(t)+Av_\varepsilon(t)\ni 0$$

and *a.e.* $x\varepsilon\Omega$, $v_\varepsilon(t,x)\varepsilon\beta(x,u_\varepsilon(t,x))$. Since u_ε is bounded in

$L^\infty((0,\infty)\times\Omega)$, by (H3), v_ε is bounded in $L^\infty((0,\infty)\times\Omega)$. There exists $\{\varepsilon_n\}$, $\varepsilon_n>0$, $\varepsilon_n\to 0+$ such that $v_{\varepsilon_n}\to v$ and $w_{\varepsilon_n}\to w$ in $L^\infty((0,\infty)\times\Omega)$. As in the proof of Theorem II.8 in [4], we have

$$\frac{du}{dt}=w \text{ in } \mathscr{D}'(0,\infty;L^2(\Omega))$$

and by (2), $\|\frac{du}{dt}(t)\|_\infty\leq\|A\beta u_0\|_\infty$.

Let j: $\Omega\times R\to(-\infty,\infty]$ be a proper lower semi-continuous convex function such that $a.e.\,x\varepsilon\Omega$, $\partial j(x,.)=\beta(x,.)$. We define J: $L^2(\Omega)\to(-\infty,\infty]$ by

$$J(z)=\begin{cases}\int_\Omega j(z) & \text{if } j(z)\varepsilon L^1(\Omega)\\ +\infty & \text{otherwise}\end{cases}$$

for every $z\varepsilon L^2(\Omega)$. Then J is a proper lower semi-continuous convex function on $L^2(\Omega)$ into $(-\infty,\infty]$ and the subdifferential ∂J of J is the prolongation of β to $L^2(\Omega)$ ([2]). Since $u_{\varepsilon_n}\to u$, $v_{\varepsilon_n}\to v$ in $L^\infty((0,\infty)\times\Omega)$ and

(4) $$v_{\varepsilon_n}(t)\varepsilon\partial Ju_{\varepsilon_n}(t),$$

it follows that $u(t)\varepsilon D(\partial J)$ and $v(t)\ \varepsilon\partial Ju(t)\ a.e.\,t\varepsilon(0,\infty)$. Thus

(5) $$v(t)\varepsilon\beta u(t)\ \ a.e.\,t\varepsilon(0,\infty).$$

By (4), we have

(6) $$\int_\Omega v_{\varepsilon_n}(u_{\varepsilon_n}(t)-u_{\varepsilon_n}(t-\varepsilon_n))\geq J(u_{\varepsilon_n}(t)-J(u_{\varepsilon_n}(t-\varepsilon_n)).$$

By Remark 3, there exists a proper lower semi-continuous convex function ϕ defind on $L^2(\Omega)$ into $(-\infty,\infty]$ with $\partial\phi=A_2$. Let $T>0$. We define Φ: $L^2((0,T)\times\Omega)\to(-\infty,\infty)$ by

$$\Phi(z)=\begin{cases}\int_0^T\int_\Omega\phi(z) & \text{if } \phi(z)\varepsilon L^1((0,T)\times\Omega)\\ +\infty & \text{otherwise}\end{cases}$$

for every $z \varepsilon L^2((0,T)\times\Omega)$. Then Φ is a proper lower semi-continuous convex function on $L^2((0,T)\times\Omega)$ into $(-\infty,\infty]$ and the subdifferential $\partial\Phi$ of Φ is the prolongation of $\partial\phi$ to $L^2((0,T)\times\Omega)$. By (3), $w_{\varepsilon_n}(t)+A_2 v_{\varepsilon_n}(t)\ni 0$ and thus $w_{\varepsilon_n}+\partial\Phi v_{\varepsilon_n}\ni 0$. For every $z\varepsilon L^2((0,T)\times\Omega)$,

$$\Phi(z)-\Phi(v_{\varepsilon_n})\geq\int_0^T\int_\Omega(-w_{\varepsilon_n})\ (z-v_{\varepsilon_n})$$
$$=\int_0^T\int_\Omega\ (-w_{\varepsilon_n})z+\int_0^T\int_\Omega w_{\varepsilon_n}v_{\varepsilon_n}.$$

By (6),

$$\int_0^T w_{\varepsilon_n}v_{\varepsilon_n}=\int_0^T\frac{u_{\varepsilon_n}(t)-u_{\varepsilon_n}(t-\varepsilon_n)}{\varepsilon_n}v_{\varepsilon_n}$$
$$=\int_0^T\frac{J(u_{\varepsilon_n}(t))-J(u_{\varepsilon_n}(t+\varepsilon_n))}{\varepsilon_n}$$
$$=\frac{1}{\varepsilon^n}\sum_{i=1}^{k}\int_{(i-1)\varepsilon_n}^{i\varepsilon_n}(J(u_{\varepsilon_n}(t))-J(u_{\varepsilon_n}(t-\varepsilon_n)))$$
$$=\frac{1}{\varepsilon_n}(\int_{(k-1)\varepsilon_n}^{k\varepsilon_n}J(u_{\varepsilon_n}\ (T))-\int_0^{\varepsilon_n}J(u_0))$$
$$=J(u_{\varepsilon_n}(T))-J(u_0), \tag{7}$$

where $T=k\varepsilon_n$. By [2], $t\to J(u(t))$ is absolutely continuous and $a.e.\ t\varepsilon(0,T)$,

$$\frac{d}{dt}J(u(t))=\int_\Omega v(t)\frac{du}{dt}(t). \tag{8}$$

By (8),

$$\int_0^T\int_\Omega v(t)\frac{du}{dt}(t)=\int_0^T\frac{d}{dt}(u(t))=J(u(T))-J(u_0)$$
$$=J(u(T))-J(u_{\varepsilon_n}(T))+J(u_{\varepsilon_n}(T)-J(u_0). \tag{9}$$

By (7)-(9),

$$\Phi(z)\geq\Phi(v_{\varepsilon_n})+\int_0^T\int_\Omega(-w_{\varepsilon_n})z$$

$$+\int_0^T \int_\Omega v(t) \frac{du}{dt}(t)+J(u_{\varepsilon_n}(T))-J(u(T)).$$

Since $\Phi(v)\leq \lim_{\varepsilon_n\to 0+}\Phi(v_{\varepsilon_n})$ and $J(u(t))\leq \lim_{\varepsilon_n\to 0+} J(u_{\varepsilon_n}(T))$,

$$\Phi(z)\geq \Phi(v)+\int_0^T \int_\Omega \left(-\frac{du}{dt}\right)z+\int_0^T \int_\Omega v \frac{du}{dt}.$$

Hence for every $z \varepsilon L^2((0,T)\times\Omega)$,

$$\Phi(z)-\Phi(v)\geq \int_0^T \int_\Omega \left(-\frac{du}{dt}\right)(z-v).$$

Thus $-\frac{du}{dt}\varepsilon\partial\phi(v)$, that is $\frac{du}{dt}(t)+A_2v(t) \ni 0$ $a.e.t$ $\varepsilon(0, T)$. Since $v(t)\varepsilon L^\infty(\Omega)$, $\frac{du}{dt}(t)+Av(t) \ni 0$ $a.e.t\varepsilon(0, T)$.

By (5), $\frac{du}{dt}(t)+A\beta u(t) \ni 0$ $a.e.t\varepsilon(0,T)$.

3. Example.

Let $\phi: L^2(\Omega)\to[0,\infty]$ be a function with $\phi(0)=0$ satisfying:

$$(10) \quad \phi(u_1-p(u_1-p(u_1-u_2))+\phi(u_2+p(u_1-u_2)) \leq \phi(u_1)+\phi(u_2)$$

for every u_1, $u_2 \varepsilon L^2(\Omega)$ and $p\varepsilon\mathscr{C}^1(R)$ with $0<p'<1$, $p(0)=0$.

If $\phi: L^2(\Omega)\to[0,\infty]$ is a lower semi-continuous convex function with $\phi(0)=0$ satisfying (10), then $A=\partial\phi\cap L^\infty(\Omega)\times L^\infty(\Omega)$ satisfies (HC)([4]).

Let Ω be a bounded open subset of R^N with smooth bord Γ. Let $j: R\to[0,\infty]$ be a lower semi-continuous convex function with $j(0)=0$ and $\gamma=\partial j$. We define a function ϕ on $L^2(\Omega)$ by

$$(11)\qquad \phi(u)=\begin{cases}\int_\Omega \frac{1}{2}\,|\mathrm{grad}u|^2+\int_\Gamma j(u) & \text{if } u\varepsilon H^1(\Omega)\\ +\infty & \text{otherwise.}\end{cases}$$

Then $\phi: L^2(\Omega)\to[0,\infty]$ is a lower semi-continuous convex function with $\phi(0)=0$ and the subdifferential $\partial\phi$ of ϕ:

$$(12)\qquad \partial\phi=\{(u,v)\varepsilon L^2(\Omega)\times L^2(\Omega)|u\varepsilon\ H^2(\Omega),\ v=-\Delta u\ a.e.\ \text{on } \Omega \text{ and } \frac{\partial u}{\partial n}+\gamma(u)\ni 0\ a.e.\ \text{on } \Gamma\},$$

where $\frac{\partial u}{\partial n}$ is the exterior normal derivative([4]).

By [4], ϕ of (11) satisfies (10). Thus $A=\partial\phi\cap L^\infty(\Omega)\times L^\infty(\Omega)$ for $\partial\phi$ of (12) satisfies (HC) ([4]).

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FACTORIZATION OF POLYNOMIALS OVER A DIVISION RING

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Factorization of polynomials over a division ring will be considered in this short note. In fact, L.H. Rowen[3] refined Wedderburn's method [4] of splitting polynomials. Here we improve again Rowen's result on factorization of polynomials.

We start with following well known

LEMMA 1. Let D be a division rings with the center F. Then for every two-sided ideal I of $D[x]$ there is a monic polynomial $f(x)$ in $F[x]$ such that $I=f(x)D[x]$. Moreover, I is a prime ideal if and only if $f(x)$ is irreducible in $F[x]$.

PROOF. Since $D[x]$ is a principal (left and right) ideal domain, there is a monic polynomial $f(x)$ such that $I=f(x)D[x]$ of least degree. Now for d in $D, r(x)=df(x)-f(x)d$ is in I and the degree of $r(x)$ is less than that of $f(x)$. Hence $r(x)=0$ and so $f(x)$ is in $F[x]$. Straightfowardly, it can be verified that $I=f(x)D[x]$ is prime if and only if $f(x)$ is irreducible in $F[x]$.

LEMMA 2. [2, Theorem 3, p. 179] Let D be a division ring with the center F and let K be a finite algebraic extension field of F. Then there are a division ring A and two positive integers h, m such that

(a) $D\otimes_F K=\mathrm{Mat}_h(A)$.

(b) $K\subset\mathrm{Mat}_m(D)$ as an F-algebra and m is such the

smallest positive integer.

(c) $hm = \dim_F K$.

Furthermore, A is the centralizer of K in $\mathrm{Mat}_m(\mathrm{D})$.

Following [1] a right ideal $g(x)D[x]$ is *bounded* if it contains a non-zero two-sided ideal. The sum of all non-zero two-sided ideals contained in $g(x)D[x]$ is thus a two-sided ideal and is called *the bound* of $g(x)D[x]$. We say two polynomials $g_1(x)$ and $g_2(x)$ in $D[x]$ are *right similar* if $D[x]/g_1(x)D[x]$ and $D[x]/g_2(x)D[x]$ are $D[x]$-isomorphic. In this case $g_1(x)D[x]$ and $g_2(x)D[x]$ have the same bound if one of them is bounded. Moreover, $g_1(x)$ and $g_2(x)$ are also left similar. So we just say $g_1(x)$ and $g_2(x)$ are *similar* when they are right similar.

THEOREM 3. Let D be a division ring with the center F and let $p(x)$ be an irreducible monic polynomial in $F[x]$. If $p(u) = 0$ for some algebraic element u over F, then for any irreducible decomposition $p(x) = g_1(x)g_2(x)\cdots g_n(x)$ of $p(x)$ in $D[x]$ we have

(a) Every $g_i(x)$ is similar to $g_1(x)$,

(b) deg $g_1(x)$ (hence all deg $g_i(x)$) is the smallest positive integer m such that $F[u] \subset \mathrm{Mat}_m(D)$ as an F-algebra,

(c) $D[x]/p(x)\mathrm{D}[x]$ is $D[x]$-isomorphic to $\oplus \sum D[x]/g_i(x)D[x]$ and

(d) $p(x)$ is the minimal polynomial of u.

PROOF. We note that $D[x]/p(x)D[x] = D \otimes_F F[u]$ is simple Artinian. By Lemma 2, there are a division ring A and two positive integers h, m such that deg $p(\mathrm{x}) = hm$, $D[\mathrm{x}]/p(x)D[x] = \mathrm{Mat}_h(A)$, and m is the smallest posit-

ive integer so that $F[x]/(p(x))$ is F-embedded in $\mathrm{Mat}_m(D)$. Actually there is a minimal right ideal V of the simple Artinian ring $D[x]/p(x)D[x]$ with $\dim_D V=m$ and $F[x]/(p(x))$ is F-embedded in $\mathrm{End}_D(V)$.

Let $V=D[x]/\beta(x)D[x]$ with $p(x)=\alpha(x)\beta(x)$ in $D[x]$. Then since V is a minimal right ideal, $\beta(x)D[x]$ is a minimal right ideal of $D[x]$ and so $\beta(x)$ is irreducible in $D[x]$. Now for an irreducible decomposition $p(x)=\beta(x)\beta_2(x)\cdots\beta_k(x)$ in $D[x]$, it can be verified that $\beta(x)D[x]$ and $\beta_i(x)$ have $p(x)D[x]$ as the bound. (see [2], p. 39) So $\beta(x)$ and each $\beta_i(x)$ are similar. In particular, deg $\beta(x)=$deg $\beta_i(x)$ for $i=2,\ldots, k$. Moreover, since deg $\beta(x)=m$ and deg $p(x)=mk$, we have $h=k$.

Now consider the given irreducible decomposition $p(x)=g_1(x)\ldots g_n(x)$ in the assumption. Then obviously $n=k$ and each $g_i(x)D[x]$ has the bound $p(x)D[x]$. So each $g_i(x)$ is similar to $\beta(x)$. Of course deg $g_i(x)=m$ is the smallest positive integer such that $F[x]/(p(x))$ is F-embedded in $\mathrm{Mat}_m(D)$. So we prove (a) and (b).

For (c), recall that the bound of each $g_i(x)D[x]$ is $p(x)D[x]$. Since $p(x)$ is irreducible in $F[x]$, $D[x]/p(x)D[x]$ is $D[x]$-isomorphic to $\oplus\sum D[x]/g_i(x)D[x]$ by [1, Theorem 20, p. 45].

Finally for (d), let I be the ideal of polynomial $f(x)$ in $D[x]$ such that $f(u)=0$. Then $p(x)$ is in I and so I is a non-zero two-sided ideal of $D[x]$. Hence by Lemma 1 there exists a monic polynomial $f_0(x)$ in $F[x]$ such that $I=f_0(x)D[x]$. But since $p(x)$ is irreducible in $F[x]$, we have $p(x)=f_0(x)$ and so $I=p(x)D[x]$. Hence $p(x)$ is the minimal polynomial of u and the proof is completed.

Observing Theorem 3 that every irreducible factor $g(x)$ of $p(x)$ has the same degree m which is the least positive integer such that $F[u] \subset \mathrm{Mat}_m(D)$ as F- algebras, we get following immediately.

COROLLARY 4. [3, Theorem 1.5] Let D be a division ring with the center F and let $p(x)$ be an irreducible polynomial in $F[x]$. If $p(d)=0$ for some element d in D, then $p(x)$ splits into linear factors in $D[x]$ and $p(x)$ is the minimal polynomial of d.

PROOF. In this case since $F[u] \subset D$, we have $m=1$. Hence each $g_i(x)$ is linear in any irreducible decompostion of $p(x)$.

COROLLARY 5. Let D be a division ring with the center F and let $p(x)$ be an irreducible monic polynomial in $F[x]$. If deg $p(x)$ is prime, then either $p(x)$ is irreducible in $D[x]$ or $p(x)$ splits into linear factors in $D[x]$.

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ERGODIC THEOREMS FOR AN ASYMPTOTICALLY NONEXPANSIVE SEMI-GROUP IN HILBERT SPACES

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1. Introduction

The origin of ergodic theory lies in statistical mechanics. We are interested in proving the existence of limit of time averages. The recent developements in the ergodic theory of nonlinear mappings in Hilbert space started with the result of B. Baillon([1]).

Baillon considered a nonexpansive mapping T of a real Hilbert space H into itself. He prove that if T has fixed points in H then for every x in H, the cesàro mean: $S_n x=\frac{1}{n}\sum_{k=0}^{n-1} T^k x$ converges weakly as $n\to\infty$ to a fixed point of T.

A corresponding theorem for a strongly continuous one parameter semi-group of nonexpansive mappings $S(t)$, $t\geq 0$ was given soon after Baillon's work by Baillon and Brezis ([2]). Also, a similar result of Baillon's work was obtained by Hirano and Takahashi for an asymptotically nonexpansive mapping ([3]).

But above results are all the cases for existence of weak limit of cesàro mean.

From the example of Genel and Lindenstrauss ([4]), it follows that there exists a nonexpansive mapping such that the time average does not converge strongly.

Therefore, Pazy ([5]) gave some further assumes on the mapping in order to assure the strong convergence of cesàro mean.

A corresponding result for a continuous one parameter semi-group of nonexpansive mapping $S(t)$, $t\geq 0$ was proved soon after Pazy's work by J.K. Kim and K.S. Ha ([6]) and recently, J.K. Kim and K.P. Park proved the existence of strong limit of cesàro mean for an asymptotically nonexpansive mapping ([7]).

In this paper, we are going to prove the cesàro mean $A_\lambda x=\frac{1}{\lambda}\int_0^\lambda S(t)x\,dt$ converges strongly to a common fixed point $F(S)=\bigcup F(S(t))$ for an asymptotically nonexpansive semi-group $S(t)$, $t\geq 0$.

Furthermore, in the near future, we are going to study of the existence of strong limit of cesàro mean for an almost nonexpansive mapping.

2. Main results

Let H denote a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$ which is induced by inner product.

Let C be a closed convex subset of H and let $\{S(t): t\geq 0\}$ be a family of mapping from C into itself satisfying the following conditions:

(i). $S(t+s)=S(t)S(s)$ for all $t, s\geq 0$

(ii). $S(0)x=Ix$ for all $x\in C$,

(iii). $S(t)x$ is continuous in $t\geq 0$ for all $x\in C$,

(iv). $\|S(t)x-S(t)y\|\leq \alpha_t\|x-y\|$ for all $x, y\in C$,

where $\lim_{t\to\infty}\alpha_t=1$.

The family $\{S(t): t\geq 0\}$ is called an asymptotically nonexpansive semi-group on C.

Let $F(S(t))$ be the set of all fixed points of $S(t)$ in C for every $t \geq 0$ and $F(S) = \cup F(S(t))$ (common fixed point of $S(t)$, $t \geq 0$.

Let we define the cesàro mean:

$$A_\lambda x = \frac{1}{\lambda} \int_0^\lambda S(t) x dt \text{ for all } x \in C \text{ and } \lambda > 0.$$

The following theorem is well known ([3]).

Theorem 1. ([3]) Let C be a closed convex subset of a real Hilbert space H. And let $\{S(t): t \geq 0\}$ be an asymptotically nonexpansive semi-group and for all z in C, $\{S(t)z\}$ is bounded. Then the cesaro mean $\{A_\lambda x\}$ converges weakly to a common fixed point p in $F(S)$.

Let B be a unit ball of l^2 it is shown that there exists a nonexpansive mapping T and a point x in B such that the cesàro mean does not converge strongly in l^2. In this paper, we will prove that one has strong convergence of $\{A_\lambda x\}$ to a common fixed point p of $F(S)$ for an asymptotically uonexpansive semi-group $S(t)$, $t \geq 0$, adding suitable assumption.

Proposition 2. Let C and $\{S(t): t \geq 0\}$ satisfy the same assumptions as in theorem 1. Then the $F(S)$ is nonempty and it is closed and convex subset of C.

Proof. In ([3]) $F(S)$ is nonempty, closedness of $F(S)$ is nonempty, closedness of $F(S)$ is obvious. To show convexity, it is sufficient to prove that $z = (x+y)/2 \in F(S)$ for all $x, y \in F(S)$, we have

$$\|S(t)z - x\| = \|S(t)z - S(t)x\| \leq \alpha_t \|z - x\| = 1/2\alpha \|x - y\|.$$
$$\|S(t)z - y\| = \|S(t)z - S(t)y\| \leq \alpha_t \|z - y\| = 1/2\alpha \|x - y\|.$$

α_t α_t

Since Hilbert space is uniformly convex Banach space, we have

$$\|z-S(t)x\|\leq 1/2(1-\delta(2/\alpha_t))\alpha_t\|x-y\|$$

and hence for $t\geq 0$,

$$z=\lim_{s\to\infty} S(s)z=\lim_{s\to\infty} S(s+t)z$$
$$=S(t)\lim_{s\to\infty} S(s)z=S(t)z.$$

LEMMA 3. Let C and $\{S(t): t\geq 0\}$ satisfy the same assumptions as in theorem 1. Then for each x in C and $\varepsilon>0$, there exists $t_0>0$, such that for all $t\geq t_0$, there exists $\lambda_0>0$ satisfying

$$\|A_\lambda x-S(t)A_\lambda x\|<\varepsilon \text{ for all } \lambda\geq\lambda_0$$

PROOF. Since

$$\|A_\lambda x-u\|^2=\frac{1}{\lambda}\int_0^\lambda\|S(t)x-u\|^2dt$$
$$-\frac{1}{\lambda}\int_0^\lambda\|S(t)x-A_\lambda x\|^2dt$$

for all x in C, u in H.
If we set $u=S(t)A_\lambda x$, then

$$\|A_\lambda x-S(t)A_\lambda x\|^2=\frac{1}{\lambda}\int_0^\lambda\|S(s)x-S(t)A_\lambda x\|^2ds$$
$$-\frac{1}{\lambda}\int_0^\lambda\|S(s)x-A_\lambda x\|^2\, ds$$

for all $t<\lambda$. Hence

$$\|A_\lambda x-S(t)A_\lambda x\|^2=\frac{1}{\lambda}\int_0^\lambda\|S(s)x-S(t)A_\lambda x\|^2\, ds$$
$$+\frac{1}{\lambda}\int_t^\lambda\|S(s)x-A_\lambda x\|^2\, ds$$
$$-\frac{1}{\lambda}\int_0^\lambda\|S(s)x-A_\lambda x\|^2\, ds$$
$$=\frac{1}{\lambda}\int_0^{\lambda-t}\|S(s)x-S(t)A_\lambda x\|^2\, ds$$

$$+\frac{1}{\lambda}\int_0^{\lambda-t}\|S(s+t)x-S(t)A_\lambda x\|^2\,ds$$
$$-\frac{1}{\lambda}\int_0^{\lambda}\|S(s)x-A_\lambda x\|^2\,ds$$
$$\leq\frac{1}{\lambda}\int_0^{t}\|S(s)x-S(t)A_\lambda x\|^2\,ds$$
$$+(\alpha_t)^2\frac{1}{\lambda}\int_0^{\lambda-t}\|S(s)x-A_\lambda x\|^2\,ds$$
$$-\frac{1}{\lambda}\int_0^{\lambda}\|S(s)x-A_\lambda x\|^2\,ds$$
$$\leq\frac{1}{\lambda}\int_0^{\lambda}\|S(s)x-S(t)A_\lambda x\|^2\,ds$$
$$+(\alpha_t{}^2-1)\frac{1}{\lambda}\int_0^{\lambda-t}\|S(s)x-A_\lambda x\|^2\,ds.$$

If we set d is diameter of $\{S(s)x\colon s>0\}$, then we have $\|S(s)x-A_\lambda x\|^2\leq d^2$ for all $\lambda>0$. By the hypothesis, for all $\varepsilon>0$, there exists $t>0$ such that $(\alpha_t^2-1)\leq\frac{\varepsilon^2}{2d^2}$ for all $t>t_0$. Therefore

$$(\alpha_t^2-1)\frac{1}{\lambda}\int_0^{\lambda-t}\|S(s)t-A_\lambda x\|^2\,ds<\frac{\varepsilon^2}{2d^2}d^2=\frac{\varepsilon^2}{2}$$

and there exists $\lambda_0>0$ such that for all $\lambda>\lambda_0$

$$\frac{1}{\lambda}\int_0^{t}\|S(s)x-S(t)A_\lambda x\|^2\,ds<\frac{\varepsilon^2}{2}.$$

Hence

$$\|A_\lambda x-S(t)A_\lambda x\|<\varepsilon.$$

THEOREM 4. Let C and $\{S(t)\colon t\geq0\}$ satisfy the assumptions as in theorem 1. If $S(t)$ is compact for all $t\geq0$, then $A_\lambda x=\frac{1}{\lambda}\int_0^{\lambda}S(t)x\,dt$ converges strongly to a common fixed point p in $F(S)$ as $\lambda\to\infty$.

PROOF. By theorem 1, $\{A_\lambda x\}$ converges weakly to a

point p in $F(S)$. Let $\{A_{\lambda_k}x\}$ be a subsequence of $\{A_\lambda x\}$, then since for all $z \in C$, $\{S(t)z\}$ is bounded $\{A_{\lambda_k}x\}$ is bounded. Hence, by compactness of $S(t)$, there exists a subsequence $\{A_{\lambda_{k_j}}x\}$ of $\{A_{\lambda_k}x\}$ such that $S(t)A_{\lambda_k}x$ converges strongly to a point p_0 in C. Also by lemma 2.,
$\|p-p_0\| \leq \lim_{j\to\infty} \|A_{\lambda_{k_j}} x - S(t)A_{\lambda_{k_j}} x\| \to 0$, uniformly in $t \geq 0$,
thus $p=p_0$. Therefore, we have

$$\|A_{\lambda_{k_j}} x - p\| \leq \|A_{\lambda_{k_j}} x - S(t)A_{\lambda_{k_j}} x\| + \|S(t) \quad A_{\lambda_{k_j}} x - p\|$$

and also, it implies that $\{A_{\lambda_{k_j}} x\}$ converges strongly to p in $F(S)$ for all $x \in C$.

THEOR 5. Let C and $\{S(t): t \geq 0\}$ satisfy the same assumptions in theorem 1. If $(I-S(t))$ transfers closed bounded subsets of C into closed subsets of H, then for every $x \in C$, $\{A_\lambda x\}$ converges strongly to a common fixed point $p \in F(S)$ as $\lambda \to \infty$.

PROOF. We will prove that every subsequence of $\{A_\lambda x\}$ has a strongly convergent subsequence to a common fixed point of $S(t)$, $t \geq 0$. By theorem 1 $\{A_\lambda x\}$ converges weakly to a point $p \in F(S)$.

Let $\{A_{\lambda_k}x\}$ be a subsequence of $\{A_\lambda x\}$.

First case, if there exists a subsequence $\{A_{\lambda_{k_j}} x\}$ of $\{A_{\lambda_k}x\}$ such that $A_{\lambda_{k_j}} x \in F(S)$ for all integer j, then $\{A_{\lambda_{k_j}} x\}$ converges strongly to a point $p \in F(S)$. Second case, if there has no subsequence $\{A_{\lambda_{k_j}} x\}$ of $\{A_{\lambda_k}x\}$ with $A_{\lambda_{k_j}} x \in F(S)$ for some j we can assume without loss of generality that $A_{\lambda_k} x \in F(S)$ as a subsequence. Therefore $(I-S(t))A_{\lambda_k}x \neq 0$. On the other hand, let $G=$

$\overline{\{A_{\lambda_k} x;\ k=1,2,3\cdots,\}}$ (strong closure of $\{A_{\lambda_k}x\}$) then G is closed and bounded set, Hence $(I-S(t))G$ is a closed set by assumption. By lemma 3, since $(I-S(t))A_{\lambda_k}x$ converges strongly to 0, uniformly on $t\geq 0$, we have,

$$0\in\overline{(I-S(t))G}=(I-S(t))G.$$

Hence there exists an element $p\in G$ such that $(I-S(t))p=0$. Since $(I-S(t))A_{\lambda_k} x\neq 0$, $p\not\equiv\{A_{\lambda_k}x\}$. Hence p is an element of the derived set of $\{A_{\lambda_k}x\}$, and so there exists a subsequence $\{A_{\lambda_{k_j}} x\}$ of $\{A_{\lambda_k}x\}$ such that $\{A_{\lambda_{k_j}} x\}$ converges strongly to a point p in $F(S)$. This is the complete proof of the theorem.

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SOME REMARKS ON ISOLATED SINGULARITIES

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1. Introduction

Let $D(a;r)$ be the open disc with center at a and radius r in the complex plane, and $D'(a;r)$ be the punctured disc with center at a and radius r. We denote by $H(G)$ the class of all holomorphic functions in a plane open set G. The letter G will from now on denote a plane open set.

DIFINITION. *If* $a\in G$ and $f\in H(G-\{a\})$, then f is said to have an isolated singularity at the point a. If f can be so defined at a that the extended function is holomorphic in G the singularity is said to be removable.

If $a\in G$ and $f\in H(G-\{a\})$, then one of the following three cases must occur[4, p. 227]:

(a) f has a removable singularity at a.

(b) There are complex numders $c_1, \cdots, c_m$, where m is a positive integer and $c_m\neq 0$, such that

$$f(z)-\sum_{k=1}^{m}\frac{c_k}{(z-a)^k}$$

has a removable singularity at a.

(c) *If* $r>0$ and $D(a;r)\subset G$, then $f(D'(a;r))$ is dense in the plane.

In case (b), f is said to have a pole of order m at a. The function $\sum_{k=1}^{m}c_k(z\text{-}a)^{-k}$, a polynomial in $(z-a)^{-1}$, is

called the principal part of f at a. In case (c), f is said to have an essential singularity at a.

In this note, we investigate some properties of isolated singularities. In section 2 we find simple conditions on f that are equivalent to the statement that f has a removable singularity at a (and similarly for poles and essential singularities). In section 3 we consider the extended complex plane C_∞.

2. Isolated singularities.

We begin with the following theorem.

THEOREM 1. *If* $a \in G$ and $f \in H(G-\{a\})$, then the following statements are equivalent:

(a) f has a removable singularity at a.

(b) $f(z)$ approaches a finite limit as $z \to a$.

(c) $\lim\limits_{z \to a} (z-a)f(z)=0$.

(d) The Laurent expansion of f about a has no negative powers.

PROOF. (a) implies (b): Let g be the holomorphic extention of f. Since g is continuous at a, it follows that

$$\lim_{z \to a} f(z)=\lim_{z \to a} g(z)=g(a);$$

hence $f(z)$ approaches a finite limit as $z \to a$.

(b) implies (c): Obvious.

(c) implies (d): The function g defined in G by

$$g(z)=\begin{cases}(z-a)\ f(z) & if\ z \neq a, \\ 0 & if\ z=a,\end{cases}$$

is continuous in G and holomorphic in $G-\{a\}$. Then it

follows from a theorem [2, p. 31] that $g \in H(G)$. Thus

$$g(z) = \sum_{n=1}^{\infty} c_n (z-a)^n \quad (z \in D\,(a; r) \subset G).$$

Consequently we have

$$f(z) = \sum_{n=0}^{\infty} c_{+1}(z-a)^n \quad (z \in D'(a; r) \subset G).$$

(d) implies (a): Let $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ be its Laurent expansion in $D'(a; r) \subset G$. Then the function g defined in G by

$$g(z) = \begin{cases} f(z) & if\ z \neq a, \\ c_0 & if\ z = a, \end{cases}$$

is holomorphic in G and agrees with f in $G - \{a\}$.

We consider now the characterization of poles.

THEOREM 2. *If* $a \in G$ and $f \in H(G - \{a\})$, then the following statements are equivalent:

(a) f has a pole at a.

(b) $\lim_{z \to a} f(z) = \infty$.

(c) There is a positive integer m and a $g \in H(G)$ with $g(a) \neq 0$ such that $f(z) = (z-a)^{-m}\, g(z)$.

(d) There is a positive integer m such that $(z-a)^m f(z)$ approaches a finite nonzero limit as $z \to a$.

(e) The Laurent expansion of f about a has a positive but finite number of negative powers.

PROOF. (a) implies (b): Let $\sum_{k=1}^{m} c_k\, (z-a)^{-k}$ be the principal part of f at a and let g be the holomorphic extension

of $f(z-a)^{-k}$ · Since $c_m \neq 0$ and g is continuous at a, it follows that

$$\lim_{z \to a} f(z) = \lim_{z \to a} \sum_{k=1}^{m} c_k (z-a)^{-k} + g(z) = \infty.$$

(b) implies (c): Let M be a positive real number. Since $\lim_{z \to a} f(z) = \infty$, there exists $r > 0$ such that $D(a;r) \subset G$ and $|f(z)| \geq M$ whenever $z \in D'(a;r)$. Then $1/f \in H(D'(a;r))$ and $\lim_{z \to a} [f(z)]^{-1} = 0$. Hence, $h(z) = [f(z)]^{-1}$ for $z \neq a$ and $h(a) = 0$, is holomorphic in $D(a;r)$. However, since $h(a) = 0$ it follows that $h(z) = (z-a)^m h_1(z)$ for some $h_1 \in H(D(a;r))$ with $h_1(a) \neq 0$ and some integer $m \geq 1$. Define $g(a) = 1/h_1(a)$, and $g(z) = (z-a)^m f(z)$ in $G - \{a\}$. Then $g \in H(G)$, $f(z) = (z-a)^{-m} g(z)$, and $g(a) \neq 0$.

(c) implies (d) : Obvious.

(d) implies (e) : *If* $(z-a)^m f(z)$ approaches a finite nonzero limit, then $(z-a)^m f(z)$ has a removable singularity at a by THEOREM 1.

Hence there exists $r > 0$ such that $D(a;r) \subset G$ and

$$(z-a)^m f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad (c_0 \neq 0, z \in D'(a;r)),$$

so we find upon dividing by $(z-a)^m$ that the Laurent expansion of f about a has a positive but finite number of negative powers.

(e) implies (a) : Let $f(z) = \sum_{n=-m}^{\infty} c_n (z-a)^n$ ($c_{-m} \neq 0$) be its

Laurent expansion in $D'(a;r) \subset G$. Define $g(a) = c_0$, and $g(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ in $D'(a;r)$. Then $g \in H(D(a;r))$, and hence

has a removable singularity at a.

For a more detailed discussion of isolated singularities, we consider the conditions

$$(A) \lim_{z \to a} |z-a|^s f(z) = 0,$$

$$(B) \lim_{z \to a} |z-a|^s f(z) = \infty,$$

where s is some real number.

LEMMA 3. Let f have an isolated singularity at a and suppose $f \not\equiv 0$. If either (A) or (B) holds for some real numder s, then there is an integer m such that (A) holds if $s > m$ and (B) holds if $s < m$; furthermore, f has a removable singularity at a if $m \leq 0$ and has a pole at a if $m > 0$.

PROOF. If (A) holds for a certain s, then it holds for all larger s, and hence for some integer p. Then $(z-a)^p f(z)$ has a removable singularity at a. Suppose $f \in H(D'(a;r))$, and let g be the holomorphic extension of $(z-a)^p f(z)$. Since $g(a)=0$ and $g \not\equiv 0$, there exists a unique positive integer k such that

$$g(z) = (z-a)^k g_1(z) \quad (z \in D(a;r))$$

where $g_1 \in H(D(a;r))$ and $g_1(a) \neq 0$. Hence we have

$$(1) \lim_{z \to a} |z-a|^s f(z) = \lim_{z \to a} |(z-a)^{s+k-p} g_1(z)|$$

$$=\begin{cases} 0 & \text{if } s > p-k \\ \infty & \text{if } s < p-k. \end{cases}$$

Thus (A) holds for all $s > m = p-k$, while (B) holds for all $s < m$.

Assume now that (B) holds for some s; then it holds for all smaller s, and hence for some integer n. By Theorem 2, there is a positive integer p and a $g_2 \in H(D(a; r))$ with $g_2(a) \neq 0$ such that

$$(z-a)^n f(z) = (z-a)^{-p} g_2(z).$$

Put $m = n+p$. Then we have

$$(2)\ \lim_{z \to a} |z-a|^s |f(z)| = \lim_{z \to a} |(z-a)^{s-m} g_2(z)|$$

$$=\begin{cases} 0 & \text{if } s > m \\ \infty & \text{if } s < m. \end{cases}$$

Finally, suppose $m \leq 0$. Then, by (1) and (2), we have

$$\lim_{z \to a} (z-a) f(z) = \lim_{z \to a} (z-a)^{1-m} g_i(z) = 0 \quad (i=1,2),$$

and hence f has a removable singularity at a. *If* $m > 0$, then

$$\lim_{z \to a} f(z) = \lim_{z \to a} (z-a)^{-m} g(z) = \infty \quad (i=1,2).$$

Hence f has a pole at a.

THEOREM 4. *If* $a \in G$ and $f \in H(G - \{a\})$, then the following statements are equivalent:

(a) f has an essential singularity at a.

(b) f(z) does not approach a finite or infinite limit as $z \to a$.

(c) The Laurent expansion of f about a has an infinite number of negative powers.

(d) Neither $\lim_{z\to a}|z-a|^s|f(z)|=0$ nor $\lim_{z\to a}|z-a|^s|f(z)|=\infty$ holds for any real number s.

(e) To each complex number w there corresponds a sequence $\{z_n\}$ such that $z_n\to a$ and $f(z_n)\to w$ as $n\to\infty$.

Proof. The equivalence of (a), (b) and (c) follows from Theorem 1 and Theorem 2. By Lemma 3, (a) implies (d). And (d) implies (a), by Theorem 1 and Theorem 2. Thus it suffices to show that (a) is equivalent to (e).

Suppose (a) holds. Choose $\delta>0$ such that $D(a;1/\delta)\subset G$. Then $D(a;1/\delta+n)\subset G$ for $n=1,2,\cdots$. Choose $w_n\in f(D'(a;1/\delta+n))\cap D(w;1/n)$, and choose $z_n\in D'(a;1/\delta+n)$ such that $f(z_n)=w_n$. Then $z_n\to a$ and $f(z_n)\to w$; hence (e) holds.

Conversely, assume that (e) holds. Suppose $D(a;r)\subset G$. Let U be a nonempty open set, and choose a point $w\in U$ with $w\neq f(a)$. Choose $\delta>0$ such that $D(w;\delta)\subset U$. Let $\{z_n\}$ be a sequence such that $z_n\to a$ and $f(z\ )\to w$. (Since $w\neq f(a)$, $z_n\neq a$ for infinitely many n). Then there exists an integer N such that $0<|z_N-a|<r$ and $|f(z_N)-w|<\delta$. Thus $f(z_N)\in f(D'(a;r))\cap U$. Since U is anarbitrary open set, it follows that $f(D'(a;r))$ is dense.

3. The extended complex plane.

For many purpose it is useful to extend the system **C** of complex numbers by introduction of a symbol ∞ to represent infinity. For any $r>0$, let $D'(\infty;r)$ be the set of all complex numbers z such that $|z|>r$, put $D(\infty;r)=D'(\infty;r)\cup\{\infty\}$. The set $\mathbf{C}_\infty=\mathbf{C}\cup\{\infty\}$ is topologized in

the following manner:

DEFINITION. A subset of C_∞ is open if and only if it is the union of discs $D(a;r)$, where the a's are arbitrary points of C_∞ and the r's are arbitrary positive numbers.

THEOREM 5. Let τ be the topology as in the above definition. Then $U\in\tau$ if and only if U is an open subset of $\mathbf{C}$ or C_∞-U is a closed compact subset of $\mathbf{C}$. That is, the set C_∞ with the topology τ is the one point compactification of $\mathbf{C}$.

PROOF. Suppose $U\in\tau$. If $\infty\notin U$, it is clear that U is an open subset of $\mathbf{C}$. Suppose $\infty\in U$, and let $U=\bigcup D\ (a;r)$. *If* $a\in\mathbf{C}$, then $D(a;r)^c$ is a closed subset of $\mathbf{C}$. And $D(a;r)^c$ is a closed bounded subset of $\mathbf{C}$ if $a=\infty$. Consequently,

$$C_\infty-U=\bigcap_{a\neq\infty} D(a;r)^c \cap \bigcap_{a=\infty} U(a;r)^c$$

is a closed bounded subset of $\mathbf{C}$; hence $C_\infty-U$ is a closed compact subset of $\mathbf{C}$.

Conversely, suppose that U is an open subset of $\mathbf{C}$ or $C_\infty-U$ is a closed compact subset of $\mathbf{C}$. If U is an open subset of $\mathbf{C}$, it is clear that $U\in\tau$. If $C_\infty-U$ is a closed compact subset of $\mathbf{C}$, then $C_\infty-U$ is a bounded subset of $\mathbf{C}$. Thus there exists $r>0$ such that $|z|\leq r$ for every $z\in C_\infty-U$, and so $D(\infty;r)\subset U$. On the other hand, $\mathbf{C}-(C_\infty-U)=\mathbf{C}\cap U$ is an open subset of $\mathbf{C}$. Hence $\mathbf{C}\cap U=\bigcup D\ (a;r_a)$, and so

$$U=(\mathbf{C}\cap U)\cup\{\infty\}=\bigcup D(a;r_a)\bigcup D(\infty;r).$$

Consequently $U\in\tau$.

We note that the extended complex plane C_∞ is homeomorphic to a sphere. In fact, a homeomorphism φ of C_∞

onto the unit sphere (where equation in three-dimensional space is $x_1^2 + x_2^2 + x_3^2 = 1$) can be explicitely exhibited : put $\varphi(\infty)=(0,0,1)$. and put

$$\varphi(z) = (2x/|z|^2+1,\ 2y/|z|^2+1,\ |z|^2-1/|z|^2+1)$$

for all complex numbers $z=x+iy$ [1, p. 18; 3, p. 9]. φ is called a stereographic projection.

The behavior of a complex function f at ∞ may be studied by considering $\tilde{f}(z) = f(1/z)$ at 0. It is clear that $f\in H(D'(\infty;r))$ if and only $\tilde{f}\in H(D'(0;1/r))$. The formal definitions are as follows;

Definition. If f is holomorphic in a punctured disc $D'(\infty;r)$, we say that f has an isolated singularity at ∞. We say that f has a removable singularity, a pole, or an essential singularity at ∞ if $\tilde{f}$ has, respectively, a removable singularity, a pole, or, an essential singularity at 0.

Theorem 6. Let f be an entire function. Then

(a) f has a removable singularity at ∞ if and only if it is constant.

(b) f has a pole at ∞ of order m if and only if it is a polynomial of degree m.

(c) f has an essential singularity at ∞ if and only if it is not a polynomial.

Proof. (a) It is clear that every constant function has a removable singularity at ∞. Conversely, suppose that f has a removable singularity at ∞. Since $\tilde{f}$ has a removable singularity at 0, $\tilde{f}(z)$ approaches a finite limit as

$z \to 0$. We define $f(\infty)$ to be this limit, and we thus see that f is entire on $\mathbf{C}_\infty$. Since $\mathbf{C}_\infty$ is compact, f is bounded. Hence, by Liouville's theorem, f is constant.

(b) Suppose f has a pole of order m. Then $\tilde{f}(z) - \sum_{k=1}^{m} c_k z^{-k} (c_m \neq 0)$ has a removable singularity at 0; hence $g(z) = f(z) - \sum_{k=1}^{m} c_k z^k$ has removable singularity at ∞. Since g is entire, it follows from (a) that g is constant. Thus f is a polynomial of degree m. Conversely, suppose that $f(z) = \sum_{k=0}^{m} c_k z^k$ $(c_m \neq 0)$ is a polynomial of degree m. Then

$$h(z) = z^m f(z) = c_m + c_{m-1}\ z + \cdots + c_0 z^m$$

is an entire function and $h(0) = c_m \neq 0$. Hence f has a pole at ∞ of order m, by Theorem 2.

(c) Immediate from (a) and (b).

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ON SEMI-CLOSURE STRUCTURES AND TOPOLOGICAL MODIFYING STRUCTURES

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1. Semi-closure structures

Let X be any non empty set and $\mathscr{P}(X)$ the power set of X. A function $u:\mathscr{P}(X)\to\mathscr{P}(X)$ is called a semi-closure structure [3] on X if satisfies the following four conditions;

1) $u(\phi)=\phi$,
2) $A\subset u(A)$ for each $A\in\mathscr{P}(X)$.
3) $A\subset B \Rightarrow u(A)\subset u(B)$ for each $A, B \in\mathscr{P}(X)$, and
4) $u(A)=u(u(A))$ for each $A\in\mathscr{P}(X)$.

A pair (X,u) where u is a semi-closure structure on X, is called a semi-closure space. These concepts are generalizations of the more familiar Kuratowski closure operator and topological spaces, respectively. For a convinience, we shall agree to use $\mathscr{U}$ as $\{A\subset X|\ u(X-A)=X-A\}$. Clearly, a semi-closure structure u is satisfied i) $X,\phi \in \mathscr{U}$ and ii) for every $A\in\mathscr{U}$ $i\in I$, $\bigcup_{i\in I} A\in\mathscr{U}$, but the finite intersection of elements of $\mathscr{U}$ is not an element of $\mathscr{U}$, in general (A family $\mathscr{U}$ of subsets of X satisfying the above conditions i) and ii) is called a pretopology [1], a supratopology [2], or a semi-topology [6] for X.).

The concept of semi-closure structures is motivated by the following examples;

EXAMPLE 1.1. Let $(X, \mathscr{T})$ be a topological space and - and 0 denote the closure operator and the interior operator in X, respectively. Then

$$\mathscr{T}^{\alpha}=\{A\subset X | A\subset A^{0-}\},$$
$$\mathscr{T}^{\beta}=\{A\subset X | A\subset A^{-0}\}, \text{ and}$$
$$\mathscr{T}^{\gamma}=\{A\subset X | A\subset A^{-0-}\}$$

are pretopologies but not topologies for X[1,5].

EXAMPLE 1.2[**6**]. Let X and Y be any two non empty sets and $\mathscr{F}$ a subcollection of $\{f | f: X\to Y \text{ is a function}\}$. Let $K(f,g)$ denote the coincidence set of f and g, consisting of all points $x\in X$ such that $f(x)=g(x)$. Define $u:\mathscr{P}(X)\to\mathscr{P}(X)$ by

$$u(A)=\cap\{K(f,g) \mid K(f,g) \supset A, f, g\in\mathscr{F}\}.$$

Then if $\bigcap_{f,g\in F} K(f,g)=\phi$, then u is a semi-closure structure on X.

Moreover, if $Y=\{0,1\}$, then the above semi-closure stucture u is a Kuratowski closure operator.

EXAMPLE 1.3 [8]. Let X be any non empty set and G and $\mathscr{E}$ denote a transformation group of X and the equivalence relations of X, respectively. Then between the complete lattice $\mathscr{G}$ (the set of all subgroups of G) and the complete lattice $\mathscr{E}$ there can be established a dual (inverse) Galois connection[7] $\mathscr{G}\underset{\tau}{\overset{\sigma}{\rightleftarrows}}\mathscr{E}$ such that

1) $\sigma(A)=\{a\sim b | f(a)=b, \text{ for some } f\in A\}$ for each subgroup A of G and

2) $\tau(\sim)=\{f\in G | f(x)\sim x, \text{for any } x\in X\}$ for each $\sim\in\mathscr{E}$.

By the Galois connection $(\sigma\tau, \tau\sigma)$, we can prove that if

$\sigma(\phi)=\phi$ and $\tau(\phi)=\phi$, then $\sigma\tau$ and $\tau\sigma$ are semi-closure structures on $\mathscr{G}$ and $\mathscr{E}$, respectively. In [7], these structures $\sigma\tau$ and $\tau\sigma$ are called closure operators.

2. Topological modifying structures

Let X be a non empty set and let Γ be a collection of semi-closure structures on X[3, 4]. Γ is called a topological modifying structure on X if for each $A, B \varepsilon \mathscr{P}(X)$ and for each $u, v \varepsilon \Gamma$, there exists an element w in Γ such that $u(A)\bigcup v(B)\supset w(A\bigcup B)$. Let (X, u) be a semi-closure space. We let $\Phi_u(x)=\{A\subset X: x \notin u(A^c)\}$. In a topological space (X, u), $\Phi_u(x)$ is clearly the neighborhood system at x in (X, u) for each $x \varepsilon X$.

Remark. (1) If a topological modifying structure Γ on X has only one element u, then u satisfies the Kuratowski closure axioms. From now on, we shall agree to use u as the unigue topology for X determined by u.

(2) Any collection of semi-closure structures on X is not a topological modifying structure on X, in general, as shown by the following example A.

(3) Any collection of topologies for X which has at least two elements is not a topological modifying structure on X, in general, as shown by the following example B.

Example A. Let $(X, \mathscr{T})$ be a topological space and and denote the closure operator and the interior operator in X, respectively. Then

$$\mathscr{T}^{\alpha}=\{A\subset X:\ A\subset A^{0-}\},$$

$$\mathscr{T}^{\beta}=\{A\subset X:\ A\subset A^{-0}\} \text{ and}$$

$$\mathscr{T}^{\gamma}=\{A\subset X:\ A\subset A^{-0}\}$$

are pretopologies [1, 5], but not topologies for X. If u, v, and w are semi-closure structures on X determined by $\mathscr{T}^{\alpha}$, $\mathscr{T}^{\beta}$, and $\mathscr{T}^{\gamma}$, respectively, then $\Gamma=\{u, v, w\}$ is not a topological modifying structure on X.

EXAMPLE B. Let $X=\{a, b, c\}$ and $u=\{X, \phi, \{a\}, \{a, b\}\}$ and $v=\{X, \phi, \{b\}, \{b, c\}\}$. Then u and v are topologies for X and $u\{c\} \cup v\{a\}=\{a, c\} \not\supset u\{a, c\}=v\{a, c\}=\{a, b, c\}$. Therefore $\Gamma=\{u, v\}$ is not a topological modifying structure on X.

THEOREM 2.1. Let Γ be a topological modifying structure on a set X. Then $\bigcup_{u \in \Gamma} \Phi_u(x)$ is a neighborhood system at x, for each $x \in X$. That is, Γ determines a topology T_Γ for X.

PROOF. 1) Set $N_x=\bigcup_{u \in \Gamma} \Phi_u(x)$ and let $A \in N_x$. Then there is $u \in \Gamma$ such that $x \notin u(A^c)$. Since $A^c \subset u(A^c)$, $x \notin A^c$ and thus $x \in A$.

2) Let A and B be two elements of N_x. Then there are $u, v \in \Gamma$ such that $x \notin u(A^c)$ and $x \notin v(B^c)$. Since Γ is a topological modifying structure on X, there exists $w \in \Gamma$ such that $x \notin u(A^c) \cup v(B^c) \supset w(A^c \cup B^c)=w((A \cap B)^c)$. Now we have $x \notin w((A \cap B)^c)$ and thus $A \cap B \in N_x$.

3) Let $A \in N_x$ and $A \subset B \subset X$. Then there exists $u \in \Gamma$ such that $x \notin u(A^c)$. Since $A \subset B$, $B^c \subset A^c$ and $u(B^c) \subset u(A^c)$. It follows that $x \notin u(B^c)$ and thus $B \in N_x$.

4) Let $A \in N_x$. Then there exists $u \in \Gamma$ such that $x \notin u(A^c)$ and we have $x \in X-u(A^c) \subset A$. Let $B=X-u(A^c)$. Then we shall prove that i) $B \in N_x$ and ii) $A \in N_y$, for each $y \in B$, that is, for each $y \in B$, $y \notin w(A^c)$ for some $w \in \Gamma$.

i) Since u is a semi-closure structure on $X(i.e., u \varepsilon \Gamma)$, $u(B^c)=u((X-u(A^c))^c)=u(u(A^c))=u(A^c)$. Since $x \notin u(A^c)$, $x \notin u((X\text{-}u(A^c))^c)$ and thus $B=X-u(A^c) \varepsilon N_x$.

ii) Since $B \cap u\ (A^c)=\phi$, $y \notin u(A^c)$ and thus $A \varepsilon N$, for each $y \varepsilon B$.

The proof is complete.

THEOREM 2.2. Let $(X, \mathscr{T})$ be a topological space and let Γ be a collection of semi-closure structures on X such that $\mathscr{T}\ \varepsilon \Gamma$ and for each $u\ \varepsilon \Gamma, u \subset \mathscr{T}$. Then,

(1) Γ is a topological modifying structure on X.

(2) $\mathscr{T}_\Gamma = \mathscr{T}$.

PROOF. (1) Let u be the closure structure on $(X, \mathscr{T})$. For each $v, w\ \varepsilon\ \Gamma$ and for each $A, B\ \varepsilon\ \mathscr{P}(X)$,

$$u(A) \cup w(B) \supset u(A) \cup u(B) = u(A \cup B).$$

Thus Γ is a topological modifying structure on X.

(2) Let A be a neighborhood of x in $(X, \mathscr{T}_\Gamma)$. Then there exists an element v in Γ such that $x \notin v\ (A^c)$. Since $v\ (A) \supset u(B)$ for each $A\ \varepsilon\ \mathscr{P}(X)$, $x \notin u(A^c)$. Thus A is a neighborhood of x in $(X, \mathscr{T})$. Conversely, let A be a neighborhood of x in $(X, \mathscr{T})$. Then $x \notin u(A^c)$ and thus $A\ \varepsilon\ \Phi_u(x) \subset \bigcup_{v \varepsilon \Gamma} \Phi_v(x)$. Therefore A is a neighborhood of x in $(X, \mathscr{T}_\Gamma)$.

The proof is complete.

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REPRESENTING MEASURES RELATED TO ALGEBRAS

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The theory of representing measures is historically the first subject discussed in the representing theory of complex homomorphism. A major motivation for the study of it stems from an attempt to specialize the Riesz representing theorem to non—selfadjoint algebras. For the function algebra, one of the useful features of boundary problems is that each complex homomorphism of the algebra can be represented as integration with respect to a positive measure. Though there are many methods in solving boundary problems, one of the particular methods is representing measures in complex C^n space.

Recently, K. Hoffman and I. Singer discussed measures and the Silov boundary in paper[1], in particular, H.S. Bear[2,3] foreshowed methods which treat the relationship between any measures and pervasive subalgebras of the maximal function algebra and showed the structure of such measures[4]. W. Rudin[5] proved the representing measure for the ball algebra and connected with Lummer's Hardy space with respect to the annihilating measure in [6].

Now let B be the open unit ball in complex C^n space and $A(B)$ be the ball algebra of B which is the class of complex continuous functions on S (the boundary of B) and holomorphic in B, then the Hahn-Banach theorem, Riesz representing theorem and properties of the Silov

boundary yield the following facts; For any bounded linear functional ϕ on $A(B)$, there exists a complex measure m_ϕ with respect to ϕ such that

$$\phi(f)=\int_S f\,dm_\phi$$

for all f in $A(B)$ and $\|\phi\|=\|m_\phi\|$, such m_ϕ is called a representing measure for ϕ. If, moreover, it is a probability measure associated linear functional ϕ on $A(B)$, m_ϕ has the property $\phi(1)=\|\phi\|=1$. Besides the Silov boundary is the smallest compact Hausdorff space on which the algebra A can be realized as a closed separating algebra of continuous functions.

1. The uniqueness of representing measures

Let X be a compact Hausdorff space in C^n and A be the function algebra on X. In the case of greatest interest to us, X will be S and A will be $A(S)$, the restriction of the ball algebra $A(B)$ to the boundary S of B. Then we have the following facts by the consequence of the maximum modulus theorem.

PROPOSITION 1.1. Two algebras $A(S)$, $A(B)$ are isometrically isomorphic Banach algebra.

PROPOSITION 1.2. Each function algebra A on X is closed in the sup-norm topology, contains the constants and separates points on X.

Since $\phi(f)$ is non-negative for any continuous function f on B, $0\leq f\leq 1$, and $\|1-f\|\leq 1$, $|\phi(1)-\phi(f)|\leq 1$, so we claim the following fact.

LEMMA 1.3. If ϕ is a linear functional on $A(B)$ and

$\phi(1)=\|\phi\|=1$, the associated measure is a probability measure.

It follows that any representing measure for ϕ is a probability measure. In above argument we could replace B by any closed subset B' of B such that $|\phi(f)|\leq\sup_{B'}|f|$ for f in subalgebra of $A(B)$. Such a set we call a support set for ϕ, since it is a set which support a representing measure for ϕ. The difference of any representing measures for the linear functional ϕ on $A(B)$ is always orthogonal to the ball algebra $A(B)$, so the following fact is satisfied.

LEMMA 1.4. Let m_ϕ, u_ϕ be any representing measures for ϕ, then the difference of m_ϕ, u_ϕ is a real measure on B.

Though the uniqueness of representing measures is not guaranteed but we claim the following theorem by the consequence of the above facts and properties of the Silov boundary.

THEOREM 1.5. If there is no non-zero real measure on the Silov boundary which is orthogonal to the ball algebra $A(B)$, then each ϕ in the maximal ideal of $A(B)$ has a unique representing measure.

Any algebra is a Dirichlet algebra on B if and only if no non-zero real measure on B is orthogonal to $A(B)$. This implies the following fact.

THEOREM 1.6. If the ball algebra $A(B)$ is a Dirichlet algebra on its Silov boundary, then each linear functional ϕ in the maximal ideal of $A(B)$ has a unique representing measure.

PROPOSITION 1.7. Representing measures are unique for every complex homomorphism of H^p for $p=\infty$. Since H^∞ possesses a property which is very close to the Dirichlet property.

2. Some properties of M

Now M is the class of those representing measure m_ϕ on the sphere S which is the boundary of the open unit B in C^n that satisfies

$$\phi(f)=\int_S f\ dm_\phi$$

for every f in $A(B)$. When $n=1$, M has exactly one member, namely normalized Lebesgue measure on the unit circle.

LEMMA 2.1. M is a convex set and weak*-compact.

LEMMA 2.2. M also has the corresponding weak*-topology. In general, it turns out to be a very large set when $n>1$. Moreover the members of M are the circular probability measure m_θ on S, these satisfy

$$\int_S v(e^{i\theta}\zeta)\quad dm_\theta(\zeta)=\int_S v\ dm_\theta$$

for every v in the class of continuous function $C(S)$ on S for every real θ.

To see some others, take $n=2$, for simplicity. Let m_τ be any probability measure on $\bar{U}(U$: open ball in $C^2)$, then the following fact is satisfied.

PROPOSITION 2.3. For every g in $A(U)$,

$$\int_{\bar{U}} g\ dm_\tau=\phi(g).$$

For example, m_τ might be concentrated on a simple closed curve in U that surrounds the origin, in such a way that m_τ solves the Dirichlet problem at 0 relative to the domain bounded by a simple closed curve in U. Then the measure m satisfies;

THEOREM 2.4. If the measure m_τ satisfies the following equation

$$\int_S v\ dm_\phi = \int_{\bar{U}} dm_\tau(z)\frac{1}{2\pi}\cdot\int_0^{2\pi} v(z, e^{i\theta}\ \sqrt{1-|z|^2})\ d\theta,$$

then m_τ belongs to M for every $v\in C(S)$ and $z\in\bar{U}$.

PROOF. To see this, simply note that the inner integral on the right side of the theorem 2.4, with v replaced by f in $A(B)$, equals $f(z,0)$. The support of this m_ϕ is the set of all (z, w) in S for which z lies in the support of m_τ.

Furthermore the set M plays a role in the study of the Lumer's Hardy space $(LH)^p(B)$ on the open unit ball B. First we introduce the definition of this space.

DEFINITION. The Lumer's Hardy space $(LH)^p(B)$ is the class of holomorphic in B provided that $|f|^p$ has a pluri-harmonic majorant in B, i.e., provided that $|f|^p\leq Re\ g$ for some holomorphic g in B for $0<p<\infty$.

We now list some consequences of the $(LH)^p(B)$ space. Since $(LH)^p(B)$ contains a closed subspace that is isomorphic to l^∞ and lies in $H^\infty(B)$, it follows that $A(B)$ is separable in the norm topology of $(LH)^p(B)$.

PROPOSITION 2.5. (i) $(LH)^p(B)$ is not separable and $A(B)$ is dense in $(LH)^p(B)$

(ii) $(LH)^2(B)$ is not isomorphic to a Hilbert space.

To see the connection between M and $(LH)^p$, associate to every continuous real function v on S, the numbers;

$$\alpha(v)=\sup\{\int_S v\, dm_\phi:\ v\text{:real continuous on } S,\ m\in M\}$$

$$\beta(v)=\inf\ \{u(0):\ u\in Re\ A(B),\ u\geq v \text{ on } S\}.$$

Since every m_ϕ in M satisfy $\phi(f)=\int_S f\, dm_\phi$ with $Re\, f$ in place of f, it is clear that $\alpha(v)\leq\beta(v)$. The converse inequality is proved in [7]. So we have the remark as following:

REMARK 2.6. The preceding numbers $\alpha(v)$, $\beta(v)$ are equal on S.

This remark and above facts imply the following.

THEOREM 2.7. A holomorphic f in D lies in $(LH)^p(B)$ if and only if

$$\sup_{r.m}\int_S |f(r)|^p\ dm_\phi<\infty$$

where $0<r<1$, $m_\phi\in M$.

3. Representing measures on the Silov boundary

Our discussion of maximal subalgebras of $C(X)$ has not involved any detailed information about the relation of the compact Hausdorff space X to the algebra A. Further discussion requires the introduction of the maximal ideal space and Silov boundary for A.

Let A be a closed subalegbra of $C(X)$, as usual containing the constants and separating points. The space of maximal ideals of A is the set $M(A)$ of all non-zero complex linear functionals on A which are multiplicative. Each such multiplicative functional is automatically of

norm 1 and we give to $M(A)$ the weak topology which it inherits as a subset of the unit shpere in the conjugate space of A. The space $M(A)$ is the largest compact Hausdorff space on which the algebra A can be realized as a separating algebra of continuous functions.

In $M(A)$ there is a unique minimal closed subset $\Gamma(A)$ on which every function in A assumes its maximum modulus. We call $\Gamma(A)$ the Silov boundary for A[8]. Since each function in A assumes its maximum on $\Gamma(A)$ we may regard A as a subalgebra of $C(\Gamma)$. The minimality of $\Gamma(A)$ we may regard A as a of $C(\Gamma)$. The minimality of Γ shows that Γ is the smallest compact Hausdorff space on which A can be realized as a closed separating algebra of continuous functions. So we can define the representing measure m_ϕ on the Silov boundary Γ like the preceding methods as following. If $z\in M(A)$, there is a positive measure m_ϕ on Γ such that

$$f(z)=\int_\Gamma f\ dm_\phi$$

for every f in A. This representing results from the fact that any continuous linear functionals on $C(\Gamma)$ which has norm 1. To apply to the representing measure, let us make the following definition.

Definition. The algebra A is called pervasive if A is a pervasive subalgebra of $C(\Gamma)$.

It follows that if A is a pervasive subalgebra of $C(X)$ then $X=\Gamma$ but A may be pervasive on Γ and not on X.

Theorem 3.1. Let A be a pervasive subalgebra of $C(\Gamma)$, let $z\in M(A)-\Gamma$ and m_ϕ be any representing measure on Γ. Then the closed support of m_ϕ is all of Γ.

PROOF. Let K be the closed support of m_ϕ. Suppose K is a proper closed subset of Γ. Since $f(z)=\int_K f\ dm_\phi$,

$$|f(z)|\leq \sup_K |f|,$$

and since A is pervasive the measure m_ϕ defines a multiplicative linear functional on $C(K)$. Thus m_ϕ must be a point mass, which is absurd since $z\notin\Gamma$.

COROLLARY 3.2. Let A be a pervasive subalgebra of $C(\Gamma)$ and let f be a function in A which has norm 1. If there is a point $z\in M(A)-\Gamma$ such that $|f(z)|=1$, then f is a constant.

PROOF. Choose a representing measure m_ϕ. Since m_ϕ has mass 1, $|f|\leq 1$, and

$$1=|f(z)|=|\int_\Gamma f\ dm_\phi|,$$

it is clear that $f(x)=f(z)$ for all x in Γ.

Of course Theorem 3.1 and its corollary hold for essential maximal algebra. We have stated them for pervasive algebra to emphasize once again that the pervasive property of maximal algebras is the fundamental one.

PROPOSITION 3.3. Let f be a function in A which has norm 1, and let K be the subset of $M(A)$ on which $f=1$. Let A_K be the algebra obtained by restricting A to the set K. Then A_K is closed and

i) $M(A_K)=K$

ii) If $z\in K$, then any representing measure m_ϕ is support on $K\cap\Gamma$.

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ON EXISTENCE OF FIXED POINTS IN 2-METRIC SPAECS

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In this paper, several fixed point theorems for a pair of mappings of a (S, T)-orbitally complete 2-metric space into itself are proved. In [2] and [6], L.B. Ciric and R.E. Smithson introduced the concepts of orbital completeness and orbital continuity of mappings on metric spaces and K. Iseki extended the concepts of orbital completeness and orbital continuity of mappings on metric spaces and K. Iseki extended the concepts of orbital completeness and orbital continuity to 2-metric spaces ([3]). Especially L.B. Ciric proved the following theorem ([2]): Let (X, d) be a T-orbitally complete metric space and a mapping T of X into itself be orbitally continuous. If T satisfies the following condition:

$$\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(y, Tx)\} \leqslant \alpha d(x, y)$$

for every x, y in X and for some α $(0<\alpha<1)$, then for each x_0 in X, the sequence $\{T^n x_0\}$ converges to a fixed point of T.

Motivated by this result of L.B. Ciric, J. Achari ([1]), K. Iseki ([4]) and S.N. Mishra ([5]) extended this result to multivalued mappings on metric spaces and 2-metric spaces, respectively.

common fixed point in X.

PROOF. For given x_0 in X, consider the orbit of S and T at the point x_0, $0(x_0, S, T)=\{x_0=x,\ x_1=Sx_0,\ x_2=Tx_1, \cdots,\ x_{2n}=Tx_{2n-1},\ x_{2n+1}=Sx_{2n}, \cdots\cdots\}$. If, for some m, $x_m=x_{m+1}$, then S and T have a common fixed point x_m in X. Thus we suppose that $x_m \neq x_{m+1}$ for all m=1, 2, 3, From the condition (A), for $x=x_{2n}$ and $y=x_{2n+1}$, we have

$$\begin{aligned}
&\min\{d(Sx_{2n}, Tx_{2n+1}, z),\ d(x_{2n}, Sx_{2n}, z),\\
&\qquad d(x_{2n+1}, Tx_{2n+1}, z),\ d(Sx_{2n}, TSx_{2n}, z),\\
&\qquad d(x_{2n+1}, TSx_{2n}, z)\}\\
&+k\min\{d(x_{2n}, Tx_{2n+1}, z),\ d(x_{2n+1}, Sx_{2n}, z),\\
&\qquad d(x_{2n}, STx_{2n+1}, z),\ d(Tx_{2n+1}, TSx_{2n}, z)\}\\
&\quad \leqslant \alpha d(x_{2n}, x_{2n+1}, z)
\end{aligned}$$

or

$$\begin{aligned}
&\min\{d(x_{2n+1}, x_{2n+2}, z),\ d(x_{2n}, x_{2n+1}, z)\}\\
&\quad \leqslant \alpha d(x_{2n}, x_{2n+1}, z)
\end{aligned}$$

for every non-negative integer n. Since (X, d) is a 2-metric space, $d(x_{2n}, x_{2n+1}, z) \neq 0$ for some z in X. Hence if $d(x_{2n}, x_{2n+1}, z) < d(x_{2n+1}, x_{2n+2}, z)$, then we have $d(x_{2n}, x_{2n+1}, z) \leqslant \alpha d(x_{2n}, x_{2n+1}, z)$ for $\alpha \in (0, 1)$, which is impossible and so we have $d(x_{2n+1}, x_{2n+2}, z) \leqslant \alpha d(x_2, x_{n+1}, z)$. Similarly, we have $d(x_{2n}, x_{2n+1}, z) \leqslant \alpha d(x_{2n-1}, x_{2n}, z)$. Therefore $d(x_m, x_{m+1}, z) \leqslant \alpha d(x_{m-1}, x_m, z)$ for every non-negative integer m and hence since $d(x_0, x_1, x_m) \leqslant d(x_0, x_1, x_{m-1}) + d(x_0, x_{m-1}, x_m) + d(x_{m-1}, x_1, x_m) = d(x_0, x_1, x_{m-1}) + d(x_m, x_{m-1}, x_2) + d(x_m, x_{m-1}, x_1)$, we have

$$\begin{aligned}
&d(x_0, x_1, x_m) \leqslant d(x_0, x_1, x_{m-1}) + \alpha^{m-1}(d(x_1, x_0, x_0)\\
&+d(x_1, x_0, x_1)) = d(x_0, x_1, x_{m-1})\\
&\quad \leqslant d(x_0, x_1, x_{m-2})
\end{aligned}$$

$$\begin{aligned}&\leqslant \cdots\cdots\\&\leqslant d(x_0,\ x_1,\ x_2)\\&\leqslant \alpha d(x_0,\ x_0,\ x_1)\\&=0.\end{aligned}$$

Therefore, from $d(x_m,\ x_n,\ z)\leqslant \sum_{k=0}^{m-n-2} d(x_m,\ x_{n+k},\ x_{n+k+1})+\sum_{k=0}^{m-n-1} d(x_{n+k},\ x_{n+k+1},\ z) < \frac{\alpha^n}{1-\alpha}(d(x_0,\ x_1,\ z)+d(x_0,\ x_1,\ z))$, it follows that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a (S, T)-orbitally complete, $\{x_n\}$ converges to some point u in X. Using the sequential continuity of S, we have

$$\begin{aligned}0&\leqslant d(u,\ Su,\ z)\\&\leqslant d(u,\ Su,\ x_{2n+1})+d(u,\ x_{2n+1},z)+d(x_{2n+1},\ Su,\ z)\\&=d(u,\ Su,\ Sx_{2n})+d(u,\ x_{2n+1},\ z)+d(Sx_{2n},\ Su,\ z)\\&\to 0 \text{ as } n\to\infty.\end{aligned}$$

Therefore $d(u,\ Su,\ z)=0$ for all z in X. Thus u is a fixed point of S. Similarly, u is also a fixed point of T, that is, u is a common fixed point of S and T. Next, let $k>\alpha$ and to prove the uniqueness of a common fixed point of S and T, let u and v be common fixed points of S and T. Since (X, d) is a 2-metrics pace, there exists a point w in X such that $d(v,\ u,\ w)\neq 0$, For this w in X,

$$\begin{aligned}&\min\{d(Sv,\ Tu,\ w),\ d(v,\ Sv,\ w),\ d(u,\ Tu,\ w),\\&\qquad d(Sv,\ TSv,\ w),\ d(u,\ TSv,\ w)\}\\&+k\min\{d(v,\ Tu,w),\ d(u,\ Sv,\ w),\ d(v,\ STu,w),\\&\qquad d(Tu,\ TSv,\ w)\}\\&\quad\leqslant \alpha d(v,\ u,\ w)\end{aligned}$$

gives

Before going our main theorems, we need the following definitons:

DEFINITION 1. In a 2-metric space (X, d), if $d(x_n, x, z)$ converges to 0 for all z in X as $n \to \infty$, we say that a sequence $\{x_n\}$ in X converges to x and x is called the limit of this sequence.

DEFINITION 2. In a 2-metric space (X, d), if $d(x_m, x_n, z)$ converges to 0 for all z in X as $m, n \to \infty$, we say that a sequence $\{x_n\}$ in X is called a Cauchy sequence in X. If every Cauchy sequence is convergent, X is called complete.

DEFINITION 3. Let S and T be two mappings from a 2-metric space (X, d) into itself. For any x_0 in X, a sequence $0(x_0, S, T) = \{x_0 = x,\ x_1 = Sx_0,\ x_2 = Tx_1, \cdots,\ x_{2n} = Tx_{2n-1},\ x_{2n+1} = Sx_{2n}, \cdots\cdots\}$ is called an orbit of S and T at the point x_0 in X.

DEFINITION 4. A 2-metric space (X, d) is (S, T)-orbitally complete if every Cauchy sequence contained in the orbit of S and T at some point converges in X.

If $S = T$ in Definition 3, then $0(x_0, S, T)$ is called an orbit of T at the point x_0 in X, and a 2-metric space (X, d) in which every Cauchy sepuence contained in the orbit $0(x_0, T, T) = 0(x_0, T)$ converges is called T-orbitally complete.

REMARK 1. Every complete 2-metric space is (S, T)-orbitally complete but the converse is not true. For example, Let $X = (0, 1] \times (0, 1]$ and define a 2-metric d on X by

$$d(x,y,z)=\frac{1}{2}\text{abs}\begin{vmatrix} x_1 & x_2 & 1 \\ y_1 & y_2 & 1 \\ z_1 & z_2 & 1 \end{vmatrix}$$

for every $x=(x_1, x_2)$, $y=(y_1, y_2)$ and $z=(z_1, z_2)$ in X. Let $T(x, y)=(\frac{x+1}{2}, \frac{y+1}{2})$ and $S(x, y)=(x, y)$ for every x, y in X. Then a 2-metric space (X, d) is (S, T)-orbitally complete but not complete.

Definition 5. A mapping T of 2-metric space (X, d) into itself is said to be sequentially continuous if for every sequence $\{x_n\}$ such that $d(x_n, x, z)$ converges to 0 for all z in X as $n\to\infty$, $d(Tx_n, Tx, z)$ converges to 0 as $n\to\infty$. A mapping T is called orbitally continuous if for all z in X, $d(T^n x, y, z)$ converges to as $n\to\infty$ implies that $d(T^{n+1}x, Ty, z)$ converges to 0 as $n\to\infty$. Every sequentially continuous mapping is orbitally continuous.

Now we are ready to give our main theorems:

Theorem 6. Let (X, d) be a (S, T)-orbitally complete 2-metric space and mappings S and T of X into itself be sequentially continuous. If mappings S and T satisfy the following condition (A):

$$\begin{aligned}\text{(A)}\ \min\{&d(Sx, Ty, z), d(x, Sx, z), d(y, Ty, z), \\ &d(Sx, TSx, z), d(y, TSx, z)\} \\ +k\min\{&d(x, Ty, z), d(y, Sx, z), d(x, STy, z), \\ &d(Ty, TSx, z)\}\leqslant \alpha d(x,y,z)\end{aligned}$$

for every x, y, z in X, where $\alpha\in(0,1)$ and k is a real number, then the orbit of S and T at x_0, $0(x_0, S, T)$, converges to a point u in X and u is a common fixed point of S and T. If $k>\alpha$, then S and T have a unique

$$d(v,\ u,\ w)\leqslant\frac{\alpha}{k}d(v,\ u,\ w),$$

which is impossible. Therefore $u=v$. This proves that S and T have a unique common fixed point u in X.

REMARK 2. In Theorem 6, if we take $S=T$, we have a sequence $\{x_n\}$, where $x_n=T^n x_0$ such that $d(x_n,\ u,\ z)$ converges to 0 as $n\to\infty$. Since T is sequentially continuous, it is orbitally continuous. Hence we have $d(Tx_n,\ Tu,\ z)$ converges to 0 as $n\to\infty$. Thus, in Theorem 6, if $T=S$ is orbitally continuous, $\{T^n x_0\}$ converges to a fixed point u of T. If $k=-1$ and (X,d) is a bounded complete 2-metric space, this result is similar to the result of S.N. Mishra ([5]).

COROLLARY 7. Let (X,d) be a (S,T)-orbitally complete 2-metric space. If mappings S and T of X into itself satisfy the following condition (B):

$$\begin{aligned}\text{(B)}\quad &\min\{d(Sx,\ Ty,\ z),\ d(x\ Sx,z),\ d(y,\ Ty,\ z)\}\\ &+k\min\{d(x,\ Ty,\ z),\ d(y,\ Sx,\ z)\}\\ &\leqslant\alpha d(x,\ y,\ z)\end{aligned}$$

for every $x,\ y,\ z$ in X, where $\alpha\in(0,1)$ and k is a real number, then the orbit of S and T at x_0, $0(x_0,\ S,\ T)$, converges to a point u in X and u is a common fixed point of S and T. If $k>\alpha$, then S and T have a unique common fixed point in X.

We note that in Corollary 7 if we put $S=T$ and $k=-1$ and if (X,d) is a bounded complete 2-metric space and T is orbitally continuous, then we have the result of K. lseki ([3]), which extends the result of L.B. Ciric([2]) to a 2-metric space.

For any positive integer powers of S and T, we have the following theorem from Theorem 6:

THEOREM 8. Let (X,d) be a (S,T)-orbitally complete 2-metric space and mappings S and T of X into itself be sequentially continuous. If mappings S and T satisfy the following condition (C):

$$\begin{aligned}
\text{(C)}\quad & \min\{d(S^s x,\ T^t y,\ z),\ d(x,\ S^s x,\ z),\ d(y,\ T^t y,\ z),\\
&\qquad d(y,\ T^t S^s x,\ z),\ d(y,\ T^t S^s x, z)\}\\
&+k\min\{d(x,\ T^t y,\ z),\ d(y,\ S^s x,\ z),\ d(x,\ S^s T^t y,\ z),\\
&\qquad d(T^t y,\ T^t S^s x,\ z)\}\\
&\leqslant \alpha d(x,\ y,\ z),
\end{aligned}$$

for every $x,\ y,\ z$ in X and for some positive integers s and t, where $\alpha \in (0,1)$ and k is a real number such that $k > \alpha$, then S and T have a unique common fixed point in X.

PROOF. If we take $P = S^s$ and $Q = T^t$, by Theorem 6, P and Q have a unique common fixed point u in X, that is, $Pu = Qu = u$. From this,

$$Su = PSu = QSu \text{ and } Tu = PTu = QTu,$$

that is, Su and Tu are common fixed points of P and Q. If we put $x = Su$ and $y = Tu$ in the condition (C), we obtain $d(Su, Tu, z) \leqslant \alpha d(Su,\ Tu,\ z)$ for $\alpha \in (0,1)$, which means $Su = Tu$. Therefore the uniqueness of u implies that $u = Su = Tu$, that is, u is a unique common fixed point of S and T.

THEOREM 9. Let (X,d) be a (S,T)-orbitally complete 2-metric space and $\{S_n\}$ and $\{T_n\}$ be sequences of sequentially continuous mappings of X into itself and let $\{s_q\}$

and $\{t_p\}$ be sequences of positive integers such that for any positive integers pair p, q $(p\neq q)$,

$$\min\{d(S_p^{sp}(x),\ T_q^{tq}(y),\ z),\ d(x,\ S_p^{sp}(x),\ z),$$
$$d(y,\ T_q^{tq}(y),\ z),\ d(S_p^{sp}(x),\ T_q^{tq}S_p^{sp}(x),\ z),$$
$$d(y,\ T_q^{tq}S_p^{sp}(x),\ z)\}$$
$$+k\min\{d(x,\ T_q^{tq}(y),\ z),\ d(y,\ S_p^{sp}(x),\ z),$$
$$d(x,\ S_p^{sp}T_q^{tq}(y), z),\ d(T_p^{tp}(y),\ T_q^{tq}S_p^{sp}(x), z)\}$$
$$\leqslant \alpha d(x,y,z),$$

for every x,y,z in X, where $\alpha\in(0,1)$ and k is a real number such that $k>\alpha$. Then the sequences $\{S_n\}$ and $\{T_n\}$ have a unique common fixed point u in X.

PROOF. If we take any pair of positive integers pair p and q $(p\neq q)$, then, by Theorem 8, S_p and T_q have a unique common fixed point in X. Since p and q are arbitrary, this theorem follows.

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A NOTE ON WEAKLY IRRESOLUTE MAPPINGS

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I. Introduction.

In 1963, N. Levine introduced a class of semi-continuous mappings which properly contains the class of all continuous mappings and in [5], the notion of an irresolute mapping which is stronger than that of semi-continuity, but is independent of that of continuity was introduced. The concept of weakly irresolute mappings was introduced in [2, Definition 6]. In this note, it will be shown that the class of weakly irresolute mappings properly contains that of irresolute mapping [5], and it is independent of that of semi-continuous mappings [6], of that of almost irresolute mappings [8] and of that of set-s-connected mappings [3]. Further, characterizations and some basic properties of weakly irresolute mappings are investigated.

Throughout this paper, spaces mean topological spaces on which no separation axioms are assumed and $f:X\to Y$ denotes a mapping from a space X into a space Y. Let A be a subset of a space X. By $T(X)$, $cl_X(A)$ and $\mathrm{int}_X(A)$ (T, $cl(A)$ and $\mathrm{int}(A)$ without confusions) we will denote, respectively, the topology on X, the closure of A and the interior of $A\subset X$. A set A is semiopen [6] in a space X if there exists an $0 \varepsilon T(X)$ such that $0\subset A\subset cl(0)$, and is semiclosed [8] iff its complement is semiopen. The intersection of all the semiclosed sets containing A is called the semic-

losure [9] of A and the union of all the semiopen sets contained in $A \subset X$ is called the semi-interior [9] of A. By $SO(X)$, $\mathrm{scl}(A)$ and $\mathrm{sint}(A)$ we will denote, respectively, the family of all semiopen sets in a space X, the semi-closure of A and the semi-interior of $A \subset X$. It was shown in well-known papers that $\mathrm{int}(A) \subset \mathrm{sint}(A) \subset A \subset \mathrm{scl}(A) \subset \mathrm{cl}(A)$; $A \subset B$ implies $\mathrm{sint}(A) \subset \mathrm{sint}(B)$ and $\mathrm{scl}(A) \subset \mathrm{scl}(B)$; A is semiopen (resp. semiclosed) iff $A = \mathrm{sint}(A)$ (resp. $A = \mathrm{scl}(A)$). A set N of a space X is called a semi-neighborhood (written semi-nbd) [1] of a point $x \in X$ if there exists an $U \in SO(X)$ such that $x \in U \subset N$. It was shown that $A \in SO(X)$ iff A is the semi-nbd of each of its points [1]. A point $p \in X$ is termed a semi-limit point of A [7] iff, for each $U \in SO(X)$ containing p, $U \cap (A - \{p\}) \neq \phi$. The union of A and $\mathrm{sd}(A)$, where $\mathrm{sd}(A)$ denotes the set of all the semi-limit points of A, called the semi-drived set of A, is equal to $\mathrm{scl}(A)$. A is semiclosed iff A contains $\mathrm{sd}(A)$.

A mapping $f: X \to Y$ is said to be irresolute [5] iff for every $V \in SO(Y)$, $f^{-1}(V) \in SO(X)$ iff for each $x \in X$ and each semi-nbd V of $f(x)$, there exists a semi-nbd U of x in X such that $f(U) \subset V$. A mapping $f: X \to Y$ is said to be semi-continuous [6] iff for every $V \in T(Y)$, $f^{-1}(V) \in SO(X)$. Every irresolute mapping is semi-continuous but not conversely [5]. A space X is semi-T_2[10] iff for each pair $x, y \in X$, $x \neq y$, there exist disjoint A, $B \in SO(X)$ such that $x \in A$ and $y \in B$ iff for each pair $x, y \in X$, $x \neq y$, there exists an $U \in SO(X)$ such that $y \in U$ and $x \notin \mathrm{scl}(U)$. By a semi-clopen set we mean a set which is both semiopen and semiclosed. A space X is s-

connected [11] iff no nonempty proper subset of X is semi-clopen; hence every indiscrete space is s-connected. A subset of a space X is s-connected iff it is s-connected as a subspace of X.

II. Weakly irresolute mappings.

DEFINITION 1. A mapping $f:X\to Y$ is said to be weakly irresolute [2] if for each $x \varepsilon X$ and each semi-nbd $V\subset Y$ of $f(x)$, there exists a semi-nbd U of x such that $f(U)\subset \text{scl}(V)$.

We now give a characterization of weakly irresolute mappings.

THEOREM 1. A mapping $f: X\to Y$ is weakly irresolute iff for each $0 \varepsilon SO(Y)$, $\subset \text{sint}(f^{-1}(\text{scl}(0)))$.

PROOF. Let $x \varepsilon f^{-1}(0)$. Then $f(x) \varepsilon 0$. Thus, by Definition 1, there exists a $G \varepsilon SO(X)$ containing x such that $f(G)\subset \text{scl}(0)$. This implies $x \varepsilon G \subset f^{-1}(\text{scl}(0))$, i.e., $x \varepsilon \text{sint}(f^{-1}(\text{scl}(0)))$. Conversely, let $x \varepsilon X$ and $f(x) \varepsilon 0 \varepsilon SO(Y)$. Then $x \varepsilon f^{-1}(0)\subset \text{sint}(f^{-1}(\text{scl}(0)))$. Let $G=\text{sint}(f^{-1}(\text{scl}(0)))$. Then $f(G)\subset \text{scl}(0)$. The proof is complete.

It is quite evident that every irresolute mapping is weakly irresolute. A weakly irresolute mapping may fail to be irresolute, as shown by the following example.

EXMPLE 1. Let $X=\{a,b,c,d\}$ with $T(X)=\{\phi, X, \{d\}, \{a,c\}, \{a,c,d\}\}$ and $Y=\{p,q,r\}$ with $T(Y)=\{\phi, Y, \{p\}\}$. Then the mapping $f:X\to Y$, defined by $f(a)=p$, $f(b)=f(c)=q$ and $f(d)=r$, is, obviously, weakly irresolute but not irresolute. Note that f is also not semi-continuous.

DEFINITION 2. A space X is strongly s-regular iff, for every point $x \varepsilon X$ and every semiclosed set F of X such that $x \notin F$, there exist disjoint U, $V \varepsilon SO(X)$ such that $x \varepsilon U$ and $F \subset V$.

It can be easily shown that a space X is strongly s-regular iff for each point $x \varepsilon X$ and each $V \varepsilon SO(X)$ containing x, there exists a $U \varepsilon SO(X)$ containing x such that $\mathrm{scl}(U) \subset V$.

THEOREM 2. Let $f: X \to Y$ be a weakly irresolute mapping. If Y is strongly s-regular. Then f is irresolute and hence semi-continuous.

PROOF. Let $x \varepsilon X$ and $V \varepsilon SO(Y)$ with $f(x) \varepsilon V$. Since Y is strongly s-regular, there exists an $M \varepsilon SO(Y)$ containing $f(x)$ such that $\mathrm{scl}(M) \subset V$. Since f is weakly irresolute, there exists a $U \varepsilon SO(X)$ containing x such that $f(U) \subset \mathrm{scl}(M) \subset V$. Thus f is irresolute.

A semi-continuous mapping may fail to be weakly irresolute, as shown by the following example. Therefore, weakly irresolute mappings are, in general, independent of semi-continuities from Example 1.

EXMPLE 2. Let $X = \{a, b, c\}$ with $T(X) = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $Y = \{p, q, r\}$ with $T(Y) = \{\phi, Y, \{p\}, \{q\}, \{p, q\}\}$. Then the mapping $f: X \to Y$, defined by $f(a) = p$, $f(b) = q$ and $f(c) = r$, is semi-continuous but not weakly irresolute.

A mapping f: $X \to Y$ is said to be almost-open [13] if, for every $V \varepsilon T(Y)$, $f^{-1}(\mathrm{cl}(V))$ $(\mathrm{cl} \subset (f^{-1}(V))$. It is known that every open mapping is almost-open and a continuous and almost-open mapping is always not open.

LEMMA 1. If $f: X \to Y$ is semi-continuous and almost-open, then f is irresolute.

PROOF. It is easy to prove and is thus omitted.

From Lemma 1, we obtain that a semi-continuous mapping is weakly irresolute if it is almost-open and hence open.

THEOREM 3. If $f: X \to Y$ is weakly irresolute, then scl $(f^{-1}(V)) \subset f^{-1}(\text{scl}(V))$ for each $V \varepsilon SO(Y)$.

PROOF. It is sufficient to show that if $x \varepsilon \text{ scl}(f^{-1}(V)) - f^{-1}(V)$, then $x \varepsilon f^{-1}(\text{scl}(V))$. Suppose $x \not\in f^{-1}(\text{scl}(V))$, that is, $f(x) \not\in \text{scl}(V)$. Then there exists a $W \varepsilon SO(Y)$ such that $f(x) \varepsilon W$ and $W \cap V = \phi$. Since $V \varepsilon SO(Y)$, we have $\text{scl}(W) \cap V = \phi$. Since f is weakly irresolute, there exists an $U \varepsilon SO(X)$ containing x such that $f(U) \subset \text{scl}(W)$. Accordingly, $f(U) \cap V = \phi$. On the other hand, if $x \varepsilon$ scl $(f^{-1}(V))$ and $x \not\in f^{-1}(V)$, then $x \in \text{scl}(f^{-1}(V)$, and so we have $U \cap f^{-1}(V) \neq \phi$ so that $f(U) \cap V \neq \phi$. This means a contradiction. Therefore, $x \varepsilon f^{-1}(\text{scl}(V))$. This proves the theorem.

From Theorem 3, it is obvious that if $f: X \to Y$ is weakly irresolute, then $f(\text{scl}(f^{-1}(V))) \subset \text{scl}(V)$ for each V ε $SO(Y)$.

DEFINITION 3. A mapping $f: X \to Y$ is termed almost irresolute [8] if for each point $x \varepsilon X$ and each semi-nbd $V \subset Y$ of $f(x)$, $\text{scl}(f^{-1}(V))$ is a semi-nbd of x.

An almost irresolute mapping need not be weakly irresolute, as show by the following example.

EXMPLE 3. Let $X = Y = \{a, b, c, d\}$ with topologies, $T(X) = \{\phi, X, \{a, d\}, \{c\}, \{a, c, d\}\}$ and $T(Y) = \{\phi, Y, \{a\}, \{b,$

$c\}$, $\{a,b,c\}\}$. Then the mapping $f: X \to Y$, defined by $f(a)=f(b)=f(c)=a$ and $f(d)=b$, is almost irresolute but not weakly irresolute.

THEOREM 4. If $f:X \to Y$ is almost irresolute and $\mathrm{scl}(f^{-1}(V)) \subset f^{-1}(\mathrm{scl}(V))$ for each $V \in SO(Y)$, then f is weakly irresolute.

PROOF. For any point $x \in X$ and $V \in SO(Y)$ containing $f(x)$, we have $\mathrm{scl}(f^{-1}(V)) \subset f^{-1}(\mathrm{scl}(V))$ by hypothesis. Since f is almost irresolute, there exists a $U \in SO(X)$ such that $x \in U \subset \mathrm{scl}(f^{-1}(V)) \subset f^{-1}(\mathrm{scl}(V))$. Thus $f(U) \subset \mathrm{scl}(V)$.

The converse to Theorem 4 does not hold, in general, as shown by the following example.

EXMPLE 4. Let $X=Y=\{a,b,c,d\}$ with topologies, $T(X)=\{\phi, X, \{a\}, \{c\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$ and $T(Y)=\{\phi, Y, \{a\}, \{a,c\}\}$. Then the identity mapping i is weakly irresolute but not almost irresolute.

EXAMPLE. 4. An almost irresolute mapping $f: X \to Y$ is weakly irresolute iff $\mathrm{scl}(f^{-1}(V)) \subset f^{-1}(\mathrm{scl}(V))$ for each $V \in SO(Y)$.

PROOF. From Theorem 3 and 4.

DEFINITION 4. Let A be a subset of a space X. The weakly irresolute mapping from X onto a subspace A of X is called a weakly irresolute retraction if the restriction $f|A$ is the identity mapping on A. We call such an A a weakly irresolute retract of X.

LEMMA 2. If A is semiopen and U is open in a space X, then $A \cap U$ is semiopen in X. (Refer to [9]).

THEOREM 5. Let A be a subset of a space X and f:

$X \to A$ be a weakly irresolute retraction of X onto A. If X is T_2 then A is semiclosed in X.

PROOF. Suppose A is not semiclosed. Then there exists a semi-limit point x of A in X such that $x \varepsilon$ scl(A) but $x \notin A$. Since f is weakly irresolute retraction, $f(x) \neq x$. Since X is T_2 there exist disjoint $U, V \varepsilon T(X)$ such that $x \varepsilon U$ and $f(x) \varepsilon V$. Thus $U \cap cl_X(V) = \phi$. Also, $V \cap A \varepsilon T(A)$ and hence $V \cap A \varepsilon SO(A)$ containing $f(x)$. Let $W \varepsilon SO(X)$ with $x \varepsilon W$. Then $U \cap W \varepsilon SO(X)$ contains x, by Lemma 2, and hence $(U \cap W) \cap A \neq \phi$ because $x \varepsilon sd(A)$. Therefore, there exists a point $y \varepsilon (U \cap W \cap A)$. Since $y \varepsilon A$, $f(y) = y \varepsilon U$ and hence $f(y) \notin cl_X(V)$. This shows that $f(W) \not\subset cl_X(V)$. Now $cl_A(V \cap A) = cl_X(V \cap A) \cap A \subset cl_X(V)$. Therefore, $f(W) \not\subset cl_A(V \cap A)$ which implies $f(W) \not\subset \mathrm{scl}_A(V \cap A)$. This contradicts the hypothesis that f is weakly irresolute. Thus A is semiclosed in X.

In Theorem 5, X is necessary Hausdorff, as shown by the following example

EXMPLE 5. Let $X = \{a, b, c\}$ with an indiscrete topology and let $A = \{a, b\} \subset X$. Then the mapping $f: X \to A$, defined by $f(a) = a$, $f(b) = f(c) = b$, is weakly irresolute and $f|A$ is the identity mapping on A, that is, A is weakly irresolute retract of X. However, A is not semiclosed in X.

LEMMA 3. A mapping $f: X \to Y$ has a semiclosed graph $G(f)$ [8] if for each $x \varepsilon X$, $y \varepsilon Y$ such that $f(x) \neq y$, there exist $U \varepsilon SO(X)$ and $V \varepsilon SO(Y)$ containing x and y, resepctively, such that $f(U) \cap V = \phi$.

In view of the following example, a weakly irresolute mapping may fail to have a semiclosed graph.

EXMPLE 6. Let $X=\{a,b,c\}$ with an indiscrete topology. Then clearly, the identity mapping i: $X \rightarrow X$ is weakly irresolute, but $G(i)$ is not semiclosed.

However, we have the following.

THEOREM 6. If i:$X \rightarrow Y$ *is* weakly irresolute and Y is semi-T_2 then G(f) is semiclosed in the product space $X \times Y$.

PROOF. Let $x \varepsilon X$ and $y \varepsilon Y$ such that $y \neq f(x)$. Then there exists a $V \varepsilon SO(Y)$ containing $f(x)$ such that $y \notin \mathrm{scl}(V)$, i.e., $y \varepsilon (Y - \mathrm{scl}(V)) \varepsilon SO(Y)$. Since f is weakly irresolute, there exists an $U \varepsilon SO(X)$ containing x such that $f(U) \subset \mathrm{scl}(V)$. Consequently, $f(U) \cap (Y - \mathrm{scl}(V)) = \phi$ and so, $G(f)$ is semiclosed, by Lemma 3.

The converse to Theorem 6 need not be true as shown by the following example.

EXMPLE 7. Let $X=\{a,b,c\}$ be the space with $T(X)=\{\phi, X, \{a\}, \{b\}, \{a,b\}\}$ and $Y=\{a,b,c\}$ be the discrete space. Then the identity mapping i: $X \rightarrow Y$ has a semiclosed graph but not weakly irresolute.

LEMMA 4 [11]. A space X is not s-connected iff it is the union of two nonempty disjoint semiopen(respectively, semiclosed) sets.

THEOREM 7. The s-connectedness is invariant under weakly irresolute surjections.

PROOF. Let f: $X \rightarrow Y$ be a weakly irresolute surjection on an s-connected space X. Suppose Y is not s-connec-

ted. Then exist nonempty disjoint $V_1, V_2 \varepsilon SO(Y)$ such that $V_1 \cup V_2 = Y$. Hence $f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$ and their union is X. Since f is surjective, $f^{-1}(V_i) \neq \phi$ for $i=1,2$. By Theorem 1, $f^{-1}(V_i) \subset \mathrm{sint}(f^{-1}(\mathrm{scl}(V_i))$. Since V_i is semiclosed, $f^{-1}(V_i) \subset \mathrm{sint}(f^{-1}(V_i))$. Hence, $f^{-1}(V_i) \varepsilon SO(X)$ for $i=1,2$. This means that X is not s-connected. Contradict.

EXMPLE 8. Let $X=Y=\{a,b,c\}$. Let X be the indiscrete and Y be the space with $T(Y)=\{\phi, Y, \{a,c\}, \{b,c\}, \{c\}\}$. Let $f: X \to Y$ be given by $f(a)=$ a and $f(b)=f(c)=b$. Then f is weakly irresolute, X is s-connected, but $f(X)=\{a,b\}$, is not s-connected. This example shows that the image of an s-connected set under a weakly irresolute mapping is not necessarily s-connected.

III. Weakly irresolute mappings and set-s-connected mappings.

LEMMA 5. [3]. A mapping f: $X \to Y$ is set-s-connected iff for each semi-clopen subset B of $f(X)$, $f^{-1}(B)$ is semi-clopen in X.

The following examples 9 and 10 show that the notion of weakly irresolute mappings is independent of that of set-s-connected mappings.

EXMPLE 9. Let $X=\{a,b,c\}$ with $T(X)=\{\phi, X, \{a\}, \{a, b\}, \{a,c\}\}$. Then the mapping f: $X \to X$, defined by $f(a)=f(c)=b$ and $f(b)=c$, is weakly irresolute but not set-s-connected.

EXMPLE 10. Let $X=\{a,b,c,d\}$ with $T(X)=\{\phi, X, \{a, c\}, \{d\}, \{c\}, \{c,d\}, \{a,c,d\}\}$ and $Y=\{a,b,c\}$ with $T(Y)=$

$\{\phi, Y, \{c\}, \{b\}, \{b,c\}\}$. Then the mapping $f: X \rightarrow Y$, defined by $f(a)=f(d)=a$ and $f(b)=f(c)=c$, is set-s-connected but not weakly irresolute.

THEOREM 8. If $f: X \rightarrow Y$ is weakly irresolute surjection, then f is set-s-connected.

PROOF. Let V be any semi-clopen subset of Y. Since V is semiclosed, $\mathrm{scl}(V)=V$. Thus, by Theorem 1, $f^{-1}(V) \subset \mathrm{sint}(f^{-1}(V))$. Hence $f^{-1}(V) \varepsilon SO(X)$. Moreover, by Theorem 3, $\mathrm{scl}(f^{-1}(V)) \subset f^{-1}(V)$. Hence $f^{-1}(V)$ is semiclosed in X. Since f is surjective, Lemma 5 f is set-s-connected. It is well-known that, for every space X and each $V \varepsilon SO(X)$, $\mathrm{scl}(V) \varepsilon SO(X)$ and also $\mathrm{cl}(V) \varepsilon SO(X)$.

THEOREM 9. Let X and Y be spaces. If $f: X \rightarrow Y$ is set-s-connected surjection, then f is weakly irresolute.

PROOF. Let $x \varepsilon X$ and $V \varepsilon SO(Y)$ containing $f(x)$. Then $\mathrm{scl}(V)$ is semi-clopen in Y. Since f is set-s-connected surjection, it follows from Lemma 5 that $f^{-1}(\mathrm{scl}(V))=U$ is semi-clopen in X. Therefore, $U \varepsilon SO(X)$ containing x such that $f(U) \subset \mathrm{scl}(V)$. Hence f is weakly irresolute.

COROLLARY 2. A surjection $f: X \rightarrow Y$ is set-s-connected iff f is weakly irresolute.

PROOF. From Theorem 8 and 9.

COROLLARY 3. If $f: X \rightarrow Y$ is a set-s-cennected surjection and Y is semi-T_2 then $G(f)$ is semiclosed in the product space $X \times Y$.

PROOF. From Theorem 6 and 9. In view of Example 7,

the converse to Corollary 3 is not true. For, $G(i)$ is semiclosed, but i is not set-s-connected.

Abstract

A mapping $f: X \rightarrow Y$ is introduced to be weakly irresolute if, for each $x \varepsilon X$ and each semi-neighborhood V of $f(x)$, there exists a semi-neighborhood U of x in X such that $f(U) \subset \mathrm{scl}(V)$. It will be shown that a mapping $f: X \rightarrow Y$ is weakly irresolute iff(if and only if) $f^{-1}(V) \subset \mathrm{sint}(f^{-1}(\mathrm{scl}(V)))$ for each semiopen subset V of Y. The relationship between mappings described in [3,5,6,8] and a weakly irresolute mapping will be investigated and it will be shown that every irresolute retract of a T_2-space is semiclosed.

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OPTIMAL BOUNDARY CONTROL PROBLEMS FOR SYMMETRIC HYPERBOLIC SYSTEMS WITH CHARACTERISTIC BOUNDARIES.

SUNG KAG CHANG

1. Introduction.

A number of authors [C. 1, C. 2, L. 1, L. 2, L. 3, V. 1] known to us have studied optimal boundary control problems for hyperbolic systems in several variables *with noncharacteristic boundaries*.

But many important problems, for instance, Maxwell equations and linearized shallow water equations, have characteristic boundary conditions.

In this paper, we study control problems for hyperbolic systems with characteristic boundaries, which is different from others.

Let Ω be an open domain in R^m with smooth boundary Γ for an integer $m>1$. We consider a first order differential operator of the form

$$(1.1) \qquad A(x, \partial/\partial x)=\sum_{j=1}^{m} A_j(x) \frac{\partial}{\partial x_j}+C(x) \text{ for } x\in\Omega$$

where $A_j(x)$ and $C(x)$ are $(l+n)\times(l+n)$ smooth symmetric matrix valued functions on $\bar{\Omega}$, l and n are given positive integers.

We also require that $A_j(x)$ and $C(x)$ are constant for sufficiently large $|x|$ in $\bar{\Omega}$.

We assume the uniform characteristic boundary, that is,

(1.2) the normal matrix $N_A(x)=\sum_{j=1}^{m} A_j(x)n_j(x)$ have constant rank n for all $x\in\Gamma$ where $(n_j(x))$ are the inward normals to the boundary Γ at x in Γ.

Without loss of generality, we may assume that $N_A(x)$ have k negative real eigenvalues and $(n-k)$ positive real eigenvalues.

We now consider a mixed initial boundary value problem as

$$(1.3)\quad \begin{cases} \dfrac{\partial y}{\partial t}=A(x,\ \partial/\partial x)y+h & \text{on } [0, T]\times \Omega=Q \\ \beta(x)y=u & \text{on } [0, T]\times \Gamma=\Sigma \\ y(0)=f & \text{on } \Omega, \end{cases}$$

where $\beta(x)$ is a boundary operator which annihilates the null space of the normal matrix $N_A(x)$ at $x\in\Gamma$, $h\in L^2(Q)$, $f\in L^2(\Omega)$and $u\in L^2(\Sigma)$(*).

For the simplicity, we transform our problem into one on a half-space by using local coordinate changes and a partition of unity. Thus we may assume without loss of generality that

(1.4) $\Omega=\{x\in R^m\ |x_1>0\}$ and $\Gamma=\{x\in R^m\ |x_1=0\}$.

Then the normal matrix $N_A(x)=A_1(x)$ for x in Γ.

By smooth change of coordinates, we may assume

$$(1.5)\quad A_1=\begin{pmatrix} 0 & 0 & 0 \\ 0 & A_n^- & 0 \\ 0 & 0 & A_n^+ \end{pmatrix}, \qquad A_n=\begin{pmatrix} A_n^- & 0 \\ 0 & A_n^+ \end{pmatrix}$$

where A_n^- is a negative-definite $k\times k$ matrix and A_n^+ is positive-definite $(n-k)\times(n-k)$ matrix.

(*) $L^2(Q)=L^2(Q;R^{l+n})$, $L^2(\Omega)=L^2(\Omega;R^{l+n})$ and $L^2(\Sigma)=L^2(\Sigma;R^k)$.

Henceforth, we assume (1.4) and (1.5) without loss of generality.

For $x \in R^m$, $y \in R^{l+n}$, we partition x and y as $x=(x_1, x^1)^T$, $y=(y_0, y_-, y_+)^T$ and $y_n=(y_-, y_+)^T$ where $x^1=(x_2, \cdots\cdots x_m)^T$, $y_0=(y_1, \cdots\cdots y_1)^T$, $y_-=(y_{l+1}, \cdots\cdots\cdots y_l{}^+{}_k)^T$ and $y_+=(y_{l+k+1}, \cdots\cdots\cdots y_{l+n})^T$.

In order to be well-posed for the problem (1.3), it is well known that the boundary condition $\beta(x)y=u$ can be written in the following form;

(1.6) $\qquad y_-=N(x)y_++u$ for $\quad x \in \Gamma$,

where $N(x)$ is a smooth $k \times (n-k)$ matrix valued function on Γ which is constant for sufficiently large $|x|$ on Γ (see [M. 1]).

We partition matrices $A_j(x)$ as $\begin{pmatrix} A_j{}^{11} & A_j{}^{12} \\ A_j{}^{21} & A_j{}^{22} \end{pmatrix}$ where $A_j{}^{11}$ and $A_j{}^{22}$ are $l \times l$ and $n \times n$ square matrices respectively for $j=2, \cdots\cdots, m$, and $A_j{}^{12}=(A_j{}^{21})^T$ is $l \times n$ matrix.

Let us denote $W_{11}(iw)=\sum_{j=2}^{m} A_j{}^{11}\, iw_j$, $W_{12}(iw)=\sum_{j=2}^{m} A_j{}^{12}\, iw_j$ and $W_{22}(iw)=\sum_{j=2}^{m} A_j{}^{22}\, iw_j$ for $w=(w_2, \cdots w_m) \in R^{m-1}$, $w \neq 0$,

We assume for $W_{11}(iw)$ that

(1.7) the matrix $W_{11}(iw)$ has distinct eigenvalues (pure imaginary) for every $w \in R^{m-1}$, $w \neq 0$.

We assume, without detail, other appropriate conditions for the problem (1.3) to be well-posed in the Kreiss' sense (we refer to [M. 1])

Under appropriate assumptions [M. 1], we have the following theorem.

THEOREM 1. For any $T>0$, $f \in L^2(\Omega)$, $h \in L^2(Q)$ and $u \in L^2(\Sigma)$, the problem (1.3) has a unique strong solution y

in $L^2(Q)$ and unique strong solution $y(t)$ in $L^2(\Omega)$ at each time $t \in [0, T]$.

Moreover y_n has strong boundary value in $L^2(\Sigma)$ and the following inequality holds;

$$(1.8) \quad |y(t)|_\Omega + |y|_Q + |y_n|_\Sigma \leqslant C[|f|_\Omega + |h|_Q + |u|_\Sigma],$$

where C is a constant independent on f, h and u.

REMARK. y_0 in theorem 1 may not have boundary value at all in $L^2(\Sigma)$.

We are interested in boundary control problem. Hence here after it may be assumed that f and h are fixed, moreover let $h=0$, and $C(x)=0$.

We now are in a position to formulate optimal control problem for the system (1.3) as follows.

Suppose that F and G be given bounded, self-adjoint and positive-definite linear operators on $L^2(\Omega)$.

A quadratic functional cost J is defined as

$$(1.9) \qquad J(u) = |u|_\Sigma^2 + (y, Fy)_Q + (y(T),\ Gy(T))_\Omega$$

for $u \in L^2(\Sigma)$ and correpponding solution y to the system (1.3).

$(C.P)$ Our problem is to minimize the cost $J(u)$ over $u \in L^2(\Sigma)$.

Our main goal is to show that an optimal control u^0 exists in $L^2(\Sigma)$ and it can be synthesized as a feedback form, that is,

$$u^0(t) = BP(t)y^0(t) \quad \text{for } a.e.\ t \in [0, T]$$

where B is an unbounded linear operator on $L^2(\Omega)$ into $L^2(\Gamma)$, $P(t)$ is a Riccati operator on $L^2(\Omega)$ and y^0 is the

optimal trajectory corresponding to u^0.

We state the main results in the next section and sketch briefly their proofs in section 3.

2. Main Results

We introduce an operator A on $L^2(\Omega)$ as

$$(2.1) \qquad Ay = A(x, \partial/\partial x)y \text{ for } y \in D(A)$$

where the domain $D(A) = \{y \in L^2(\Omega) \mid Ay \in L^2(\Omega) \text{ and } y_- = Ny_+ \text{ on } \Gamma\}$. Then it is easily seen that A is closed and densely defined on $L^2(\Omega)$, furthermore it generates a strongly continuous semigroup $S(t)$ on $L^2(\Omega)$.

REMARK. The adjoint A^* of A is given by

$$(2.2) \quad A^*y = -\sum_{j=1}^{m} A_j(x) \frac{\partial}{\partial x_j} y - \sum_{j=1}^{m} \left(\frac{\partial}{\partial x_j} A_j(x)\right) y$$

for $y \in D(A^*)$ where the domain

$$D(A^*) = \{ y \in L^2(\Omega) \mid A^*y \in L^2(\Omega)$$

and $y_+ = -(A_n^+)^{-1} N^T A_n^- y_-$ on $\Gamma\}$.

It is also well-known that A^* generates the adjoint semigroup $S^*(t)$ of $S(t)$ on $L^2(\Omega)$.

In order to introduce a Dirichlet map "D" which extends boundary functions to interior functions in a certain way, we consider the following boundary value problem:

$$(2.3) \quad \begin{cases} A(x, \partial/\partial x)y = Ky \text{ on } \Omega \\ y_- = Ny_+ + u \text{ on } \Gamma \end{cases}$$

for $u \in L^2(\Gamma)$ where K is a large constant.
Then we have the following lemma.

LEMMA 2.1.

The problem (2.3) is well-posed for a sufficiently large

number $K>0$. Moreover, the following inequality holds:

(2.4) $\quad |y|_\Omega + |y_n|_\Gamma \leqslant C|u|_\Gamma$ for $u \in L^2(\Gamma)$

where C is a constant independent on u.

Once we have chosen K so that (2.3) is well-posed, we fix K. For the simplicity, we may assume $K=0$ (If $K \neq 0$, we simply translate A by K)

From lemma 2.1, we define the Dirichlet map D as:

$$Du=y \text{ if } y \text{ is the solution to (2.3).}$$

Then D is a bounded linear operator on $L^2(\Gamma)$ into $L^2(\Omega)$.

Now have the following trace operator.

LEMMA 2.2.

$$D^* A^* y = A_n^- y_-|_\Gamma \quad \text{for} \quad y \in D(A^*).$$

Let us define an operator L on $L^2(\Sigma)$ as

(2.5) $\quad (Lu)(t) = A \int_0^t S(t-s)\ D\ u(s) ds \quad$ for $0 \leqq t \leqq T$

and $u \in D(L)$ which is a subspace of $L^2(\Sigma)$.

We have a semigroup representation of the solutions to the mixed problem(1.3).

THEOREM 2.3. (Semigroup Representation).

(1) the operator L is a bounded linear operator on $L^2(\Sigma)$ into $C([O, T]; L^2(\Omega))$,

(2) the solution y to the problem (1.3) is given by

(2.6) $\quad y(t) = S(t)f - (Lu)(t) \quad$ for $0 \leqslant t \leqslant T$.

From theorem 2.3, it is easily seen that our control problem $(C.P)$ has a unique optimal control u^0 in $L^2(\Sigma)$ by standard argument (see [C. 1, L. 1]).

Let us denote $\Phi(t, s)$ the evolution operator which

describes the evolution of the optimal trajectory y^0 with initial time s, $0 \leqq s \leqq t \leqq T$ (see [C. 1, L. 1]). Then we have the following synthesis of optimal control u^0.

THEOREM 2.4. (Feedback synthesis)

The optimal control u^0 can be written in the form $u^0(t) = D^*A^*P(t)y^0(t)$ a.e. $t \in [0, T]$ where $P(t)$ is a bounded self-adjoint and positive-definite operator on $L^2(\Omega)$ which satisfies the following Riccati equation:

(R.E.1) for x, $y \in L^2(\Omega)$ and $0 \leq t \leq T$,

$$(P(t)x, y)_\Omega = \int_t^T (\Phi(s,t)x, F\,\Phi(s,t)y)_\Omega\, ds$$

$$+ \int_t^T (D^*A^*P(s)\,\Phi(s,t)x, D^*A^*P(s)\Phi(s,t)y)_\Gamma\, ds$$

$$+ (\Phi(T,t)x, G\,\Phi(T,t)y)_\Omega.$$

For a moment, we assume that

(2.7) F and G map $L^2(\Omega)$ into $D(A^*)$.

We denote $\mathscr{P}$ the class of one parameter families of operators $P(t)$ on $L^2(\Omega)$ which are self-adjoint, positive-definite and satisfy the following conditions;

(2.8.1) $D^*A^*\bar{P}(\cdot): L^2(\Omega) \to L^\infty([0, T]; L^2(\Gamma))$ are bounded and

(2.8.2) $D^*A^*\bar{P}(\cdot)S(\cdot)AD: L^2(\Gamma) \to L^2(\Sigma)$ are bounded.

REMARK. Under assumption (2.7), it is shown that the operator $P(t)$ in theorem 2.4 is in the class $\mathscr{P}$ (see [C. 1]).

Then we have the following theorem.

THEOREM 2.5. Under assumption (2.7), the operator $P(t)$ in theorem 2.4 is the unique solution in the class $\mathscr{P}$ to the following Riccati equation

(R.E.2) for $x, y \in L^2(\Omega)$ and $0 \leq t \leq T$,

$$(P(t)x, y)_\Omega = \int_t^T (S(s-t)\, x,\ FS(s-t)y)_\Omega\, ds$$

$$- \int_t^T (D^*A^*P(s)S\,(s-t)\, x,\ D^*A^*P(s)S\,(s-t)y)_\Gamma\, ds + (S(T-t)x,\ GS(T-t)y)_\Omega,$$

(R.E.3) for $x, y \in D(A)$, a.e. t in $[0, T]$,

$$\frac{d}{dt}(P(t)x, y)_\Omega = -(x, Fy)_\Omega - (P(t)Ax, y)_\Omega - (P(t)\, x, Ay)_\Omega + (D^*A^*P(t)x,\ D^*A^*P(t)y)_\Gamma$$

with terminal condition $P(T) = G$.

REMARK. without smoothness assumption (2.7), we are not sure whether $D^*A^*P(t)x$ are well-defined in L^2-sense for $x \in L^2(\Omega)$.

Suppose that the pairs F_n and G_n satisfy assumption (2.7) for all n=1, 2, …. Let $P_n(t)$ be the corresponding Riccati operators to the pairs F_n and G_n, for n=1, 2, 3, ….

Now we do not assume any smoothness for F and G. Then we have the following convergence.

THEOREM 2.6. Suppose that $F_n \to F$ and $G_n \to G$ strongly on $L^2(\Omega)$ as $n \to \infty$. Then $P_n(t) \to P(t)$ uniformly in $t \in [0, T]$, strongly on $L^2(\Omega)$ as $n \to \infty$.

Moreover, for $x, y \in L^2(\Omega)$, $0 \leq t \leq T$,

$$(P(t)x, y)_\Omega = \int_t^T (S(s-t)x,\ FS(s-t)y)_\Omega\, ds + (S(T-t)x,\ GS(T-t)y)_\Omega - \lim_{n\to\infty}\int_t^T (D^*A^*P_n(s)\,S\,(s-t)x,\ D^*A^*P_n(s)S(s-t)y)_\Gamma\, ds.$$

3. Proofs of Results

We sketch the proofs of lemmas 2.1 and 2.2 briefly, and we omit proofs of theorems 2.3, 2.4, 2.5 and 2.6 since their proofs are similar to those in [c.1].

Suppose for a moment that the coefficients $A_i(x)$ and $N(x)$ in (2.3) are frozen at the values on a boundary point $x_0{}^1 \in \Gamma$.

Then we apply Fourier transform the equation (2.3) in the tangential variables x^1, and denote the transforms of y, y_0 and y_n by $\hat{y}$, $\hat{y}_0$ and $\hat{y}_n$ respectively.
We arrive at

$$(3.1)\quad \begin{cases} A_1\ (0, x_0^1)\ \dfrac{d\hat{y}}{dx_1} = (K - W(0,\ x^0{}_1,\ iw))\hat{y} \text{ for } x_1 > 0 \\ \hat{y}_- = N(x_0^1)\hat{y}_+ \ + \hat{u} \qquad\qquad \text{for } x_1 = 0 \end{cases}$$

where $W(0,\ x^1, iw) = \begin{pmatrix} W_{11}(0, x_0^1,\ iw) & W_{12}(0, x_0^1,\ iw) \\ W^T{}_{12}(0, x_0^1,\ iw) & W_{22}(0, x_0^1,\ iw) \end{pmatrix}$

and $\quad w = (w_2, \cdots\cdots,\ w_m) \in R^{m-1},\ w \neq 0.$
By assumption (1.5) and (1.7), we have

$$(3.2)\quad \begin{cases} \hat{y}_0 = [K - W_{11}\ (0, x_0^1, iw)]^{-1} W_{12}(0, x_0^1, iw)\hat{y}_n, \quad x_1 > 0 \\ A_n \dfrac{d\hat{y}_n}{d_{x1}} = [W^T{}_{12}\ (K - W_{11})^{-1}\ W_{12} + (K - W_{22})]\hat{y}_n, \\ \qquad\qquad x_1 > 0 \\ \hat{y}_- = \ N\hat{y}_+ \ + \hat{u} \qquad \text{for } x_1 = 0. \end{cases}$$

We may rewrite (3.2) in a pseudo-differential form for the variable coefficient problem as

$$(3.3.1)\quad \hat{y}_0 = [K - \overline{W}_{11}(x, iw)]^{-1} \overline{W}_{12}\ (x, iw)\hat{y}_n\ , x_1 > 0$$

$$(3.3.2)\quad A_n(x)\ \frac{d\hat{y}_n}{dx_1} = [\overline{W}^T{}_{12}\ (x, iw)(K - \overline{W}_{11}(x, iw))^{-1}\overline{W}_{12}$$

$$(x, iw) + (K - \overline{W}_{22}(x, iw))]\hat{y}_n,\ x_1 > 0$$

(3.3.3) $\hat{y}_- = N(x^1)\hat{y}_+ + \hat{u},\ x_1 = 0$ where

$\overline{W}_{11}(x, iw)$, $\overline{W}_{12}(x, iw)$ and $\overline{W}_{22}(x, iw)$ are the pseudo-differential operators of order 1 corresponding to the difierential operators $\sum_{j=2}^{m} A_j^{11}(x)\frac{\partial}{\partial x_j}$, $\sum_{j=2}^{m} A_j^{12}(x)\frac{\partial}{\partial x_j}$ and $\sum_{j=2}^{m} A_j^{22}(x)\frac{\partial}{\partial x_j}$ respectively.

Let $\overline{M}(K, x, iw) = A_n^{-1}[\overline{W}^T{}_{12}\ (K-\overline{W}_{11})^{-1}\ \overline{W}_{12} + K - \overline{W}_{22}]$.

Then we arrive at

(3.4.1) $\hat{y}_0 = (K-\overline{W}_{11})^{-1}\overline{W}_{12}\hat{y}_n \qquad x_1 > 0$

(3.4.2) $\dfrac{d\hat{y}_n}{dx_1} = \overline{M}\hat{y}_n \qquad x_1 > 0$

(3.4.3) $\hat{y}_- = N\hat{y}_+ + \hat{u} \qquad x_1 = 0.$

The problems

(3.4.2) and (3.4.3) are the same kind Majda and Osher studied in [M.1].

That is, they showed that there exists a symmetrizer of $\overline{M}$ whose symbol $\overline{R}\ (K, x, iw)$ is of order zero and satis fies the following properties (see [M.1]);

(3.5.1) $\overline{R}$ is Hermitian,

(3.5.2) $v^T\overline{R}v \geqq \delta|v|^2 - \varepsilon|g|^2$ for all vectors satisfying the boundary condition $v_- = N\ v_+ + g$,

(3.5.3) Re $(\overline{R}\overline{M}) \geqq \delta$ where $\delta > 0$ and $\varepsilon > 0$ are constants independent on $x \in \Omega$, $w \in R^{m-1}$ and $K > 0$ large enough.

Thus combining (3.4.2) and (3.4.3) with (3.5.1), (3.5.2) and (3.5.3), we have, for sufficiently large $K > 0$,

(3.6) $|y_n|_\Omega + |y_n|_\Gamma \leqq C|u|_\Gamma$ where C is a constant independent on u.

On the other hand, we suppose that y is a solution to (2.3).

We take the inner product (2.3) with y on Ω.
Then we have

$$(3.7) \qquad (A(x,\partial/\partial x)y,\ y)_\Omega = K|y|^2_\Omega .$$

By Green's formula, the left hand side of (3.7) becomes

$$-\frac{1}{2}(A_n\ y_n,\ y_n)_\Gamma - \frac{1}{2}(y, \sum_{j=1}^{m} (\frac{\partial}{\partial x_j} A_j)y)_\Omega .$$

That is,

$$K|y|^2{}_\Omega + \frac{1}{2}(y, \sum_{i=1}^{m} (\frac{\partial}{\partial x_j} A_j)y)_\Omega = -\frac{1}{2}\ (A_n\ y_n,\ y_n)_\Gamma .$$

Thus, for sufficiently large $K>0$, we arrive at

$$(3.8) \qquad |y|_\Omega \leq C\ |y_n|_\Gamma \text{ for a constant } C.$$

From (3.6) and (3.8), we derive the ineguality $|y|_\Omega + |y_n|_\Gamma \leq C|u|_\Gamma$ for sufficiently large $K>0$.
Once we have the energy inequality, we can deduce the uniqueness and existence easily as in [C. 1].
This completes the proof of lemma 2.1.

We assume, without loss of generality, $K=0$.
From Green's formula, we have for $y \in D(A^*)$ and $g \in L^2(\Gamma)$,

$$(3.9) \qquad (A^*y, Dg)_\Omega = (y, A(x, \frac{\partial}{\partial x})\ Dg)_\Omega + (A_n y_n, (Dg)_n)_\Gamma .$$

By the definition of the operator D, (3.9) becomes

$$(A^*y, Dg)_\Omega = (A_n y_n,\ (Dg)_n\ )_\Gamma .$$

From the fact that $y \in D(A^*)$ and $(Dg)_- = N(Dg)_+ + g$ on Γ, we arrive at

$$(A^*y, Dg)_\Omega = (A_n y_-, g)_\Gamma$$

which implies lemma 2.2..

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EQUIVALENT CONDITIONS IN TOTALLY UMBILICAL SUBMANIFOLDS OF A KAHLERIAN MANIFOLD

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0. Introduction

In the past seventies, Ako[1], Blair and Ludden([2]-[4]), Yano([2], [3], [26]-[33]), Okumura([4], [17], [18]), Chung[5], Eum[6], Ki([5]-[13], [31]-[33]), Kim [11], Pak([12], [13],[19]), Kwon[14], Lim and Choe[15], and Shin([20]-[22]), studided a structure induced on a hypersurface of an almost contact manifold or a submanifold of codimension 2 of an almost complex manifold. During the last 10 years, in spite of their vigorous efforts, the investigation about the submanifold of codimension 2 of a Kählerian manifold was not carried out successfully.

In 1967, Okumura[17] proved the following theorem A and in 1971, Ki[7] extended the theorem A under some conditions.

THEOREM A. *Let M be a complete, connected 2n-dimensional totally umbilical submanifold with non-zero mean curvature vector* μ *of a* $(2n+2)$*-dimensional Kählerian manifold.*

(*) *Suppose that for any tangent vector X to M the covariant derivatives of the mean curvature vector* $\tilde{\nabla}_X H$

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be tangent to M. Then M is isometric with a shere S^{2n} *of radius* $\frac{1}{\sqrt{\mu}}$.

In the above theorem, we consider the two following problems.

[PROBLEM 1] What condition is equivalent to the above (*)?

[PROBLEM 2] Under what condition does (*) exists?

The purpose of the present paper is devoted to solve the above two problems.

In I, we recall a submanifold of codimension 2 of a Kählerian manifold and find the structure equations.

In II, we find some equivalent conditions in totally umbilical submanifolds.

In III, We study submanifolds with normal (f, g, u, v, λ)-structure.

In the last section IV, we determine the submanifold of condimension 2 in a locally Fubinian manifold.

I. Submanifolds of codimension 2 of Kāhlerian manifold [8].

Let $\widetilde{M}^{2n+2}$ be a $(2n+2)$-dimensional Kählerian manifold covered by a system of coordinate neighborhoods $\{U; y^\kappa\}$, where here in the sequel the indices $\kappa, \lambda, \mu, \nu$ run over the range $\{1, 2, \cdots, 2n+2\}$, and let $(F_\mu{}^\kappa, G_{\mu\lambda})$ be the Kählerian structure of $\widetilde{M}^{2n+2}$, that is

$$(1.1) \qquad F_\mu{}^\kappa F_\lambda{}^\mu = -\delta_\lambda^\kappa$$

and $G_{\mu\lambda}$ a Riemannian metric such that

$$(1.2) \qquad G_{\beta\alpha} F_\mu{}^\beta F_\lambda{}^\alpha = G_{\mu\lambda},$$

$$\tilde{\nabla}_\mu F_\lambda{}^\kappa = 0 \tag{1.3}$$

where $\tilde{\nabla}$ denotes the operator of covariant differentiation with respect to the Christoffel symbols $\widetilde{\left\{ {\kappa \atop \mu\lambda} \right\}}$ formed with $G_{\mu\lambda}$.

Let M^{2n} be a $2n$-dimensional differentiable manifold which is covered by a system of coordinate neighborhoods $\{U; x^h\}$, where here and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, 2n\}$, and, which is differentiably immersed in $\tilde{M}^{2n+2}$ a sa submanifold of codimension 2 by the equation $y^\kappa = y^\kappa(x)$.

We put

$$B_i{}^\kappa = \partial_i y^\kappa, \quad (\partial_i = \partial/\partial x^i)$$

then $B_i{}^\kappa$ is, for each i, a local vector field of $\tilde{M}^{2n+2}$ tangent to M^{2n} and the vector fields $B_i{}^\kappa$ are linearly independent in each coordinate neighborhood. $B_i{}^\kappa$ is, for each κ, a local 1-form of M^{2n}.

We assume that we can choose two mutually orthogonal unit vectors C^κ and D^κ of $\tilde{M}^{2n+2}$ normal to M^{2n} in such a way that $2n+2$ vectors $B_i{}^\kappa, C^\kappa, D^\kappa$ give the positive orientation of $\tilde{M}^{2n+2}$.

Thus we can put

$$\begin{cases} F_\lambda{}^\kappa B_i{}^\lambda = f_i{}^h B_h{}^\kappa + u_i C^\kappa + v_i D^\kappa, \\ F_\lambda{}^\kappa C^\lambda = -\mu^i B_i{}^\kappa + \lambda D^\kappa, \\ F_\lambda{}^\kappa D^\lambda = -v^i B_i{}^\kappa - \lambda C^\kappa. \end{cases} \tag{1.4}$$

where $f_i{}^h$ is a tensor field of type (1.1) and u_i, v_i are 1-forms and λ is globally defined function on M^{2n}, u^i and v^i being defined respectively by

$$u^i = u_t g^{ti}, \quad v^i = v_t g^{ti},$$

g_{ji} being the Riemannian metric on M^{2n} induced from that of $\widetilde{M}^{2n+2}$.

From the equation (1.2) and (1.4), we have

$$(1.5) \quad \begin{cases} f_j{}^t f_t{}^h = -\delta_j{}^h + u_j u^h + v_j v^h, \\ f_j{}^t f_i{}^s g_{ts} = g_{ji} - u_j u_i - v_j v_i, \\ f_i{}^t u_t = \lambda v_i \text{ or } f_t{}^h u^t = -\lambda v^h, \\ f_i{}^t v_t = -\lambda u_i \text{ or } f_t{}^h v^t = \lambda u^h, \\ u_t u^t = v_t v^t = 1 - \lambda^2, \ u_t v^t = 0. \end{cases}$$

Putting $f_{ji} = f_j{}^t g_{ti}$, we can easily find that f_{ji} is skew-symmetric, that is, M^{2n}admits an (f, g, u, v, λ)-structure.

We denote by $\left\{ {h \atop ji} \right\}$ and ∇_i the Christoffel symbols formed with g_{ji} and the operator covariant differentiation with respect to $\left\{ {h \atop ji} \right\}$ respectively.

Then the equations of Gauss and Weingarten are

$$(1.6) \quad \begin{cases} \nabla_j B_i{}^\kappa = \partial_j B_i{}^\kappa + \widetilde{\left\{ {\kappa \atop \mu\lambda} \right\}} B_j{}^\mu B_i{}^\lambda - B_h{}^\kappa \left\{ {h \atop ji} \right\} \\ \qquad = h_{ji} C^\kappa + k_{ji} D^\kappa \\ \nabla_j C^\kappa = \partial_j C^\kappa + \widetilde{\left\{ {\kappa \atop \mu\lambda} \right\}} B_j{}^\mu C^\lambda = -h_j{}^i B_i{}^\kappa + l_j D^\kappa, \\ \nabla_j D^\kappa = \partial_j D^\kappa + \widetilde{\left\{ {\kappa \atop \mu\lambda} \right\}} B_j{}^\mu D^\lambda = -k_j{}^i B_i{}^\kappa - l_j C^\kappa \end{cases}$$

where h_{ji} and k_{ji} are second fundamental tensors of M^{2n} with respect to the normals C^κ and D^κ respectively, $h_j{}^i = h_{jt} g^{ti}$, $k_j{}^i = k_{jt} g^{tj}$ and l_j is the third fundamental tensor.

Differentiating (1.4) covariantly along M^{2n} and taking account of (1.3) and (1.6), we obtain

$$(1.7)\quad \begin{cases} \nabla_j f_i{}^t = -h_{ji}u^h + h_j{}^h u_i - k_{ji}v^h + k_j{}^h v_i, \\ \nabla_j u_i = -h_j\, f_i{}^t - \lambda k_{ji} + l_j v_i, \\ \nabla_j v_i = -k_{jt} f_i{}^t + \lambda h_{ji} - l_j u_i, \\ \nabla_j \lambda = k_{jt}u^t - h_{jt}v^t. \end{cases}$$

The *mean curvature vector field* H^κ of $\widetilde{M}^{2n+2}$ is defined by $H^\kappa = \alpha C^\kappa + \beta D^\kappa$,

where $$\alpha = \frac{1}{2n} h_t{}^t, \quad \beta = \frac{1}{2n} k_t{}^t.$$

The *mean curvature* of M^{2n} in $\widetilde{M}^{2n+2}$ is the magnitude of the mean curvature vector field, that is,

$$\mu = \alpha^2 + \beta^2.$$

If the second fundamental tensors of M^{2n} are $h_{ji} = \alpha g_{ji}$ $k_{ji} = \beta g_{ji}$, then M^{2n} is said to be *totally umbilical* [17].

II. Equivalent conditions in totally umbilical submanifolds.

In this section we assume that M^{2n} is totally umbilical submanifold of $\widetilde{M}^{2n+2}$. Then we have from (1.7)

$$(2.1)\quad \nabla_j f_i{}^h = -\alpha g_{ji} u_h + \alpha \delta_j{}^h u_i - \beta g_{ji} v^h + \beta \delta_j{}^h v^i,$$

$$(2.2)\quad \nabla_j u_i = \alpha f_{ji} - \lambda \beta g_{ji} + l_j v_i,$$

$$(2.3)\quad \nabla_j v_i = \beta f_{ji} + \lambda \alpha g_{ji} - l_j u_i,$$

$$(2.4)\quad \nabla_j \lambda = \beta u_j - \alpha v_j,$$

respectively.

First of all we prove

LEMMA 2.1. *Let M^{2n} be a $2n$-dimensional totally umbilical submanifold of a $(2n+2)$-dimensional Kählerian manifold.*

Then the following conditions are equivalent to each other;

(1) *The covariant derivative of the mean curvature vector*

$$\tilde{\nabla}_j H^\kappa \text{ is tangent to } M^{2n},$$

(2) $\nabla_k\alpha=\beta l_k,\ \nabla_k\beta=-\alpha l_k,$

(3) *The equations of Codazzi are*

$$\begin{cases}\nabla_k h_{ji}-\nabla_j h_{ki}=l_k k_{ji}-l_j k_{ki}\\ \nabla_k k_{ji}-\nabla_j k_{ki}=l_j h_{ki}-l_k h_{ji},\end{cases}$$

(4) $\nabla_k\nabla_j\lambda=-(\alpha^2+\beta^2)\ \lambda g_{kj}.$

PROOF. (1)$\Leftarrow\Rightarrow$(2) see[17].

(2)$\Rightarrow$(3): Since M^{2n} is a totally umbilical submanifold of $\widetilde{M}^{2n+2}$, we easily have

$$\begin{aligned}\nabla_k h_{ji}-\nabla_j h_{ki}&=(\nabla_k\alpha)g_{ji}-(\nabla_j\alpha)g_{ki}=(\beta l_k)g_{ji}-(\beta l_j)g_{ki}\\ &=l_k k_{ji}-l_j k_{ki}\\ \nabla_k k_{ji}-\nabla_j k_{ki}&=(\nabla_k\beta)g_{ji}-(\nabla_j\beta)g_{ki}=(-\alpha l_k)g_{ji}-(-\alpha l_j)g_{ki}\\ &=l_j h_{ki}-l_j h_{ki}.\end{aligned}$$

with the aid of (2).

(2)$\Leftarrow$(3): Transvecting g^{ji} to the equation of Codazzi, we find the equations in (2).

(2)$\Rightarrow$(4): Differentiating (2.4) covariantly along M^{2n}, and using (2.2), (2.4), we obtain

$$(2.5)\quad \nabla_k\nabla_j\lambda=-(\alpha^2+\beta^2)\lambda g_{kj}+(\nabla_k\beta+\alpha l_k)u_j-(\nabla_k\alpha-\beta l_k)v_j.$$

If the equations in (2) hold on whole M^{2n}, from(2.5) we obtain

$$\nabla_k\nabla_j\lambda=-(\alpha^2+\beta^2)\lambda g_{kj}.$$

(2)$\Rightarrow$(4): If $\nabla_k\nabla_j\lambda=-(\alpha^2+\beta^2)\lambda g_{kj}$ holds in M^{2n} we also have from (2.5)

$$(\nabla_k\beta+\alpha l_k)u_j-(\nabla_k\alpha-\beta l_k)v_j=0 \tag{2.6}$$

To begin with, let us consider the following two cases.

(CASE I) If there is some open set in $\{P\in M^{2n}|\lambda^2(p)=1\}$, then $u_i=v_i=0$ holds on the open set because of $1-\lambda^2=0$.

Differentiating these covariantly and using (2.2) and (2.3), we find

$$\alpha f_{ji}-\lambda\beta g_{ji}=0,\quad \beta f_{ji}+\lambda\alpha g_{ji}=0$$

Since f_{ji} and g_{ji} are skew-symmetric and symmetric with respect to j and i respectively, we easily verify that $\alpha=\beta=0$.

Hence the equations in (2) hold on the open set.

(CASE II) If $1-\lambda^2\neq0$ *a,e.*, on an any open set $\{P\in M^{2n}|\lambda^2(p)\neq1\}$, then transvecting (2.6) with u^j and v^j respectively yields

$$\nabla_k\beta+\alpha l_k=0,\ \nabla_k\alpha-\beta l_k=0.\quad (Q.E.D.)$$

According to the equations in (2), the mean curvature vector μ is constant.

Now, let us consider this result case by case

(i) $\mu=\alpha^2+\beta^2=0$ implies $h_{ji}=k_{ji}=0$.

(ii) $\mu\neq0$ implies $\nabla_j\nabla_i\lambda=-(\alpha^2+\beta^2)\lambda g_{ji}$,

where we have used (2.4).

Summarizing the above results (i), (ii) and Obata's theorem[16], we obtain

THEOREM 2.2. *Let M^{2n} be a ocomplete cnnected totally*

umbilical submanifold of codimension 2 of Kähelrian manifold $\widetilde{M}^{2n+2}$.

If one of the conditions in Lemma 2.1 holds on M^{2n}, *then* M^{2n} *is a to'ally geodesic submanifold or a sphere* S^{2n}.

III. Submanifold of codimension 2 with normal (f, g, u, v, λ)-structure.

We now define a tensor field N of type (1,2) as follows:

$$N_{ji}{}^h = [f, f]_{ji}{}^h + (\nabla_j u_i - \nabla_i u_j) u^h + (\nabla_j v_i - \nabla_i v_j) v^h,$$

where $[f, f]_{ji}{}^h$ is the *Nijenhuis tensor* [26] formed with $f_j{}^h$ defined by

$$[f, f]_{ji}{}^h = f_j{}^t \nabla_t f_i{}^h - f_i{}^t \nabla_t f_j{}^h - (\nabla_j f_i{}^t - \nabla_i f_j{}^t) f_t{}^h$$

The (f, g, u, v, λ)-structure is said to be *normal* ([27]-[33]) if $N_{ji}{}^h$ vanishes identically.

In this section, we assume that the totally umbilical submanifold M^{2n} with (f, g, u, v, λ)-structure is normal. Then we have

$$(3.1) \qquad (l_j v_i - l_i v_j) u^h - (l_j u_i - l_i u_j) v^h = 0$$

because of (2.1), (2.2) and (2.3).

Applying (3.1) with u_h and v^i successively, we obtain

$$(3.2) \qquad l_j (1 - \lambda^2)^2 = 0$$

with the aid of (1.5).

If there is some open set in $\{P \in M^{2n} | \lambda^2(P) = 1\}$, then we can easily verify that

$$\nabla_k \nabla_j \lambda = -(\alpha^2 + \beta^2) \lambda g_{ji}$$

because of (2.5) and (1.5).

And if $1-\lambda^2 \neq 0$ *a.e.*, then (3.2) gives $l_j=0$.

Now, differentiating (2.2) covariantly, we obtain

$$\nabla_k\nabla_j u_i - \nabla_k\nabla_i u_j = 2(\nabla_k\alpha)f_{ji} + 2\alpha(-\alpha g_{kj}u_i + \alpha g_{ki}u_j - \beta g_{kj}v_i + \beta g_{ki}v_j)$$

because of (2.1) and the fact $l_j=0$, from which, using Ricci identity,

$$(\nabla_k\alpha)f_{ji} + (\nabla_j\alpha)f_{ik} + (\nabla_i\alpha)f_{kj} = 0. \tag{3.3}$$

Transvecting (3.3) with f^{ji}, we find.

$$(\nabla_k\alpha)(n-1+\lambda^2) - \nabla_k\alpha + (u^j\nabla_j\alpha)u_k + (v^j\nabla_j\alpha)v_k = 0. \tag{3.4}$$

If we transvect u^k and v^k to this respectively, and make use of (1.5), it means

$$u^k\nabla_k\,\alpha=0,\quad v^k\nabla_k\,\alpha=0 \tag{3.5}$$

provided that $\dim M^{2n}>2$.

Substituting (3.5) into (3.4), we see that α=constant. Similarly, we can verify that β=constant from (2.3).

Therefore, taking account of (2.5), α=constant, β= constant and $l_j=0$, we have

$$\nabla_k\nabla_j\lambda = -(\alpha^2+\beta^2)\lambda g_{ki}.$$

Combining theorem 2.2 and the above result, we conclude

Theorem 3.1. *Let M^{2n} be a complete connected totally umbilical submanifold of codimension 2 of a Kahlerian manifold $\widetilde{M}^{2n+2}$.*

If the induced (f,g,u,v,λ)-structure is normal and dim $M^{2n}>2$, then M^{2n} is the same type of Theorem 2.2.

IV. Submanifold of codimension 2 in a locally Fubinian manifold

In this section, we consider a submanifold M^{2n} in a *locally Fubinian manifold*, that is, a Kählerian manifold of constant holomorphic sectional curvature. Then its curvature tensor is given by

$$(4.1)\quad \widetilde{R}_{\nu\mu\lambda\kappa}=K(G_{\nu\kappa}G_{\mu\lambda}-G_{\mu\kappa}G_{\nu\lambda}+F_{\nu\kappa}F_{\mu\lambda}-F_{\mu\kappa}F_{\nu\lambda}-2F_{\nu\mu}F_{\lambda\kappa}),$$

where K is constant(see[8],[17]).

Substituting (4.1) into the equations of Gauss, Codazzi and Ricci,

$$\widetilde{R}_{\nu\mu\lambda\kappa}B_k{}^\nu B_j{}^\mu B_i{}^\lambda B_h{}^\kappa=R_{kjih}-h_{kh}h_{ji}+h_{jh}h_{ki}-k_{kh}k_{ji}+k_{jh}k_{ki},$$
$$\widetilde{R}_{\nu\mu\lambda\kappa}B_k{}^\nu B_j{}^\mu B_i{}^\lambda C^\kappa=\nabla_k h_{ji}-\nabla_j h_{ki}-l_k k_{ji}+l_j k_{ki},$$
$$\widetilde{R}_{\nu\mu\lambda\kappa}B_k{}^\nu B_j{}^\mu B_i{}^\lambda D^\kappa=\nabla_k k_{ji}-\nabla_j k_{ki}+l_k h_{ji}-l h_{jki},$$
$$\widetilde{R}_{\nu\mu\lambda\kappa}B_k{}^\nu B_j{}^\mu C^\lambda D^\kappa=\nabla_k l_j-\nabla_j l_k+h_{kt}k_j{}^t-h_{jt}k_k{}^t,$$

we have respectively

$$(4.2)\quad \begin{aligned}&K(g_{kh}g_{ji}-g_{jh}g_{ki}+f_{kh}f_{ji}-f_{jh}f_{ki}-2f_{kj}f_{ih})\\&=R_{kjih}-h_{kh}h_{ji}+h_{jh}h_{ki}-k_{kh}k_{ji}+k_{jh}k_{ki},\end{aligned}$$

$$(4.3)\quad \begin{cases}\nabla_k h_{ji}-\nabla_j h_{ki}-l_k k_{ji}+l_j k_{ki}=K(u_k f_{ji}-u_j f_{ki}-2u_i f_{kj}),\\ \nabla_k k_{ji}-\nabla_j k_{ki}+l_k h_{ji}-l_j h_{ki}=K(v_k f_{ji}-v_j f_{ki}-2v_i f_{kj}),\end{cases}$$

$$(4.4)\quad \nabla_k l_j-\nabla_j l_k+h_{kt}k_j{}^t-h_{jt}k_k{}^t=K(v_k u_j-v_j u_k-2\lambda f_{kj}).$$

Throughout this section, we assume the submanifold M^{2n} is βg_{ji}. Then the first equation in (4.3) can be transformed into

$$(4.5)\quad (\nabla_k\alpha-\beta l_k)g_{ji}-(\nabla_j\alpha-\beta l_j)g_{ki}=K(u_k f_{ji}-u_j f_{ki}-2u_i f_{kj})$$

Transvecting g^{ji} to (4.5) and using (1.5) yields

(4.6) $$(2n-1)(\nabla_k\alpha-\beta l_k)=-3\lambda K v_k.$$

Substituting (4.6) into (4.5)×(2n−1), we easily have

(4.7) $$-3\lambda K v_k g_{ji}+3\lambda K v_j g_{ki}=(2n-1)K(u_k f_{ji}-u_j f_{ki}-2u_i f_{kj}).$$

Transvecting (4.7) with u_k and using (1.5) gives

$$3\lambda K v_j u_i=(2n-1)K\{(1-\lambda^2)f_{ji}-u_j(-\lambda v_i)-2u_i(-\lambda v_j)\}.$$

Again, transvecting $v^j u^i$ to the above equation, we also obtain

$$3\lambda K(1-\lambda^2)^2=3(n-1)K\lambda(1-\lambda^2)^2.$$

If $\lambda(1-\lambda^2)\neq 0$ *a.e.*, we have $K=(2n-1)K$, which means $K=0$, if $n>1$.

Furthermore we have from (4.6)

(4.8) $$\nabla_k\alpha=\beta l_k.$$

On the other hand, from the second equation of (4.3), we get

(4.9) $$(\nabla_k\beta+\alpha l_k)g_{ji}-(\nabla_j\beta+\alpha l_j)g_{ki}=K(v_k f_{ji}-v_j f_{ki}-2v_i f_{kj}),$$

where we have used $h_{ji}=\alpha g_{ji}$ and $k_{ji}=\beta g_{ji}$.

Transvecting g_{ji} to (4.9), we also obtain

(4.10) $$(2n-1)\ (\nabla_k\beta+\alpha l_k)=3\lambda K u_k.$$

Substituting (4.10) into (4.9)×(2n−1) yields

(4.11) $$(3\lambda K u_k)g_{ji}-(3\lambda K u_j)g_{kj}=(2n-1)K(v_k f_{ji} -v_j f_{ki}-2v_i f_{kj}).$$

Transvecting v^k to (4.11), we easily have

$$-3\lambda K u_j v_i=(2n-1)K\{(1-\lambda^2)f_{ji}-v_j(\lambda u_i)-2v_i(\lambda u_j)\}.$$

And also, transvecting u^j with the above equation gives

$$-3\lambda(1-\lambda^2)Kv_i=-3(2n-1)\lambda(1-\lambda^2)Kv_i,$$

where we have used (1.5).

Now, if $\lambda(1-\lambda^2)\neq 0$ $a.e$, then $K=0$.

This means

$$\nabla_k\beta=-\alpha l_k. \tag{4.12}$$

To begin with, let us consider the following two cases.

(Case I) If there is some open set in $\{P\in M^{2n}|\lambda(P)=0\}$, we obtain from (4.6) and (4.10)

$$\nabla_k\mu=\nabla_k(\alpha^2+\beta^2)=2(\alpha\nabla_k\alpha+\beta\nabla_k\beta)=2(\alpha\beta l_k-\alpha\beta l_k)=0.$$

This means that $\mu=\alpha^2+\beta^2$ is constant.

(Case II) If $\lambda\neq 0$ $a.e$, on any open set $\{P\in M^{2}|^{n}\lambda(P)\neq 0\}$,

then we also think the following two cases (A) and (B), that is,

(A) if $1-\lambda^2\neq 0$ $a.e.$, on the open set, from (3.8) and (3.12) we obtain that $\mu=\alpha^2+\beta^2$ is constant,

(B) if there is some open set in $\{P\in M^{2n}|(1-\lambda^2)(P)=0\}$, then $u_i=v_i=0$. Hence we deduce (4.8) and (4.12) from (4.6) and (4.10). By using the same method, we know that $\mu=\alpha^2+\beta^2$ is constant on M^{2n}. Hence, by making use of theorem 2.2, we have

Theorem 4.1. *Let M^{2n} be a complete connected totally umbilical submanifold of codimension 2 in a locally Fubinian manifold $\widetilde{M}^{2n+2}$. If dim $M^{2n}>2$, then the submanifold M^{2n} is a totally geodesic or a sphere S^{2n}.*

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ON THE JORDAN STRUCTURE IN OPERATOR ALGEBRAS

GYUNG SOO WOO

1. Introduction

The study of JB-algebras was initiated by Alfsen, Shultz and Størmer [3], even though earlier approaches have been made by von Neumann and Segal. In [3], the study of JB-algebras can be reduced to the study of Jordan algebras of self-adjoint operators on a Hilbert space and M_3^8.

The purpose of this note is to show Jordan-Banach algebra versions of some facts about C*-algebras by some modifications. In section 2, we give the formal definitions of JB-algebras and JB*-algebras and some known results. In section 3, we study projections and ideals in JB-algebra. In section 4, we study the multipliers of JB-algebras.

2. Preliminaries

A *Jordan Banach algebra* is a real Jordan algebra A equipped with a complete norm satisfying

$$||a\circ b||\leq||a||\ ||b||,\quad a,\ b\in A.$$

A *JB-algebra* is a Jordan Banach algebra A in which the norm satisfies the following two additional conditions for $a,\ b\in A$:

(i) $||a^2||=||a||^2$

(ii) $||a^2||\leq||a^2+b^2||$.

Examples of JB-algebras are JC-algebras, i.e., the norm closed. Jordan algebras of self-adjoint operators on a complex Hilbert space, and the exceptional $M_3{}^8$ consisting ofall Hermitian 3×3 matrices over the Cayley number.

Note that an associative JB-algebra can be realized as the self-adjoint part of a commutative C*-algebra [15]. In finite dimension, JB-algebras are precisely the formally real Jordan algebras. However this is not true in infinite dimensional JB-algebras [13].

A JB-algebra which is also a Banach dual space is said to be a *JBW-algebra*. Then the second dual, A^{**} of a JB-algebra A is a unital JB-algebra and moreover, it is a JBW-algebra in the Arens product which contains A [14]. A special case of this is already known; if A is a JC-algebra then A^{**} is isomorphic to a JC-algebra [12].

The reader is referred to [3, 4, 12, 13] for properties of JB-algebras. The complex analogue of JB-algebras are the JB*-algebras (Kaplansky's Jordan C*-algebras), introduced by Kaplansky, who first presented it at a lecture for the Edinburgh Mathematical Society in July 1976.

A *JB-algebra* is a complex Jordan Banach algebra $\mathscr{A}$ with an involution * such that for all $x\in\mathscr{A}$,

$||\{x,\ x^*,\ x\}||=||x||^3$ holds.

For example, every C*-algebra A is a JB-algebra in the Jordan product. The second dual, $\mathscr{A}^{**}$ of a JB*-algebra $\mathscr{A}$ with the Arens product, is a unital JB*-algebra [20].

It is known that the set of self-adjont elements of a unital JB*-algebra forms a unital JB-algebra, while, conversely, the complexification of a unital JB-algebra in a suitable norm is a JB*-algebra [17]. In [20], this also holds for non-unital JB-algebras. Therefore JB-algebras and JB*-algebras are in a one-to-one correspondence.

A *Jordan W*-algebra* is a unital JB*-algebra which is the dual of a complex Banach space. In [11], it is shown that the self-adjoint part of a Jordan W*-algebra is a JBW-algebra and the complexification of a JBW-algebra is a Jordan W*-algebra. The general theory of JB*-algebras can be found in [11, 17, 19, 20].

3. Projections and Ideals in JB-algebras

If $p^2=p$ then p is called an *idempotent*. An idempotent in a JB-algebra will be called a *projection*.

Let A be a JB-algebra and let a, b, c be elements of A. The *Jordan triple product* $\{a, b, c\}$ is defined by

$$\{a, b, c\} = (a \circ b) \circ c + a \circ (b \circ c) - (a \circ c) \circ b$$

and for $a \in A$, U_a and L_a are defined by

$$U_a b = \{a, b, a\}, \qquad L_a b = a \circ b \text{ for } b \in A.$$

Note that if A is a JC-algebra then $\{a, b, c\} = \frac{1}{2}(abc + cba)$.

Recall that two projections p and q are said to be *orthogonal* if $p \circ q = 0$.

LEMMA 3.1. Let p and q be projections in the JB-algebra A. Then the followings are equivalent.

(i) $pq=0$ (ii) $p \circ q = 0$ (iii) $\{p, q, p\} = 0$ (iv) $p+q$ is a projection.

PROOF. By [13, Lemma 4.2.2] and easy calculation.

Let A and B be JC-algebras. We call a linear map ϕ from A into B is a *Jordan homomorphism* if $\phi(a\circ b)=\phi(a)\circ\phi(b)$ for all a, $b\in A$ and ϕ takes the identity into the identity.

PROPOSITION 3.2. Let p and q be orthogonal projections of JC-algebra A and ϕ is a Jordan homomorphism. Then $\phi(\{p,x,q\})=\{\phi(p),\phi(x),\phi(q)\}$ holds for all $x\in A$.

PROOF. Since $2(p\circ x)\circ q=\{p,x,q\}$ and $\phi(p)\phi(q)=0$ by Lemma 3.1 we have

$$\begin{aligned}\phi(\{p,x,q\})&=\phi(2(p\circ x)\circ q)=2(\phi(p)\circ\phi(x))\circ\phi(q)\\&=\{\phi(p),\phi(x),\phi(q)\}.\end{aligned}$$

The following is a slight modification of [7, Proposition 1.5.8].

PROPOSITION 3.3. Let A and B be JC-algebras and ϕ is a Jordan homomorphism from A into B. If p is a projection of A, then $\phi(p)$ is a projection of B.

PROOF. We get $\{\phi(p)\}^2=\phi(p)\circ\phi(p)=\phi(p\circ p)=\phi(p^2)=\phi(p)$ since p is a projection. Hence $\phi(p)$ is a projection of B.

Recall that elements a, b in a JB-algebra A are said to *operator commute* if $L_aL_b=L_bL_a$. i.e., if $(a\circ c)\circ b=a\circ(c\circ b)$ for all c in A. If p is a projection in A than a and p operator commute if and only if $L_pa=U_pa$ or $a=U_pa+U_{e-p}a$. A projection p in A is said to be *central* if p operator commutes with every element of A.

REMARK. Central projections can be used to construct more general ideals. For example, if A is a JB-algebra, B a JB-subalgebra of A and p a central projection in A, then the set of all b in B such that $p \circ b = 0$ is an ideal in B (in fact it is a Jordan ideal). For, let $J = \{b \in B \mid p \circ b = 0\}$. If $a \in J$, $c \in B$, then $p \circ (a \circ c) = (p \circ a) \circ c = c \circ (p \circ a) = 0$. Hence $a \circ c \in J$.

A subspace J of a JB-algebra A is said to be a *Jordan ideal* in A if $L_a b \in J$ whenever $a \in J$, $b \in A$. A linear subspae J of A is a Jordan ideal if and only if $aba \in J$ whenever $a \in A$ and $b \in J$ [12]. Note that Jordan ideals correspond to two-sided ideals in the following sense; A norm closed self-adjoint complex subspace $\mathcal{J}$ of a C*-algebra $\mathcal{A}$ is a two-sided ideal if and only if its self-adjoint part $\mathcal{J}_{sa}$ is a Jordan ideal of $\mathcal{A}_{sa}$. This can be seen easily by considering the weak*-closure in $\mathcal{A}^{**}$ of $\mathcal{J}$ and using [8, Theorem 2.3], or by [12, Theorem 2].

A subspace J is said to be a *quadratic ideal* in A if $U_a b \in J$ whenever $a \in J$, $b \in A$. Note that every Jordan ideal is a quadratic ideal.

LEMMA 3.4. Let J be a Jordan ideal in a JB-algebra A. Then A/J with its natural Jordan product and quotient norm is a JB-algebra.

Let $\mathcal{A}$ be a JB*-algebra with self-adjoint part A. A Jordan ideal $\mathcal{J}$ of $\mathcal{A}$ is said to be a **-ideal* if, whenever $z \in \mathcal{J}$ then $z^* \in \mathcal{J}$. Let J be the self-adjoint part of a norm closed ideal $\mathcal{J}$ of $\mathcal{A}$, then $\mathcal{J} = J + iJ$ and J is a norm closed ideal of A.

THEOREM 3.5 [17]. Let $\mathcal{A}$ be a JB*-algebra. Let $\mathcal{J}$ be a

closed *-ideal. Then $\mathcal{A}/\mathcal{J}$, when equipped with the quotient norm, is a JB*-algebra. Furthermore, if J is the self-adjoint part of $\mathcal{J}$, then the self-adjoint part of $\mathcal{A}/\mathcal{J}$ is isometrically isomorphic to A/J.

REMARK. The self-adjoint part of Jordan *-ideals is precisely the Jordan ideal in the unital JB-algebra A which is the self-adjoint part of $\mathcal{A}$.

LEMMA 3.6. If A is a JB-algebra, then every weak *-ideal J of A^{**} is of the form $U_p(A^{**})$ for a central projection $p \in A^{**}$.

PROOF. By [3, Lemma 9.1] J will contain an increasing approximate identity $\{U_\alpha\}$, i.e., $0 \leq U_\alpha \leq 1$, $\alpha \leq \beta$ implies $U_\alpha \leq U_\beta$ and $||U_\alpha \circ a - a|| \to 0$ for all $a \in J$. Since $A^{**} = \tilde{A}$, A^{**} is monotone complete; Let p be the least upper bound of $\{U_\alpha\}$ in A^{**}. Then by [3, Theorem 3.10], $U_\alpha \to p$ strongly. It follows that $p \in J$ and $p^2 = p$ is an identity for J and this is also the greatest projection in J. Since J is an ideal,

$$U_p(A^{**}) \subseteq J = U_p(J) \subseteq U_p(A^{**}),$$

which shows $J = U_p(A^{**})$. Furthermore, if $s^2 = 1$ and $s \in A^{**}$, then $U_s p$ is a projection in J and so $U_s p \leq p$. Since $U_s^2 = I$, by positivity of the map U_s, we have

$$p = U_s^2 p \leq U_s p \leq p \quad \text{so} \quad U_s p = p.$$

Since this holds for every symmetry, by [3, Lemma 5.3] p is central.

THEOREM 3.7. If p is a central projection in a JB-algebra A, then $U_p A$ is a Jordan ideal in A. Conversely, if p is

a projection in A such that $U_p A$ is a Jordan ideal then p is central.

PROOF. If p is a central projection in A and then, for $a \in A$, $L_p a = U_p a$. Therefore, for $b \in U_p A$,

$$b \circ a = L_b a = L_b L_p a = L_p L_b a = U_p (b \circ a) \text{ and } b \circ a \in U_p A.$$

It follows that $U_p A$ is a Jordan ideal. Conversely, if $a \in A$ we must have $p \circ a \in U_p A$, thus $U_p(p \circ a) = p \circ a$. This implies $U_p a = L_p a$. Hence p is central.

4. Multipliers of JB-algebras

The concept of the multiplier algebra of a C*-algebra has been extended to JB-algebra by Edwards [10]. An element b in a second dual A^{**} of a JB-algebra A is said to a *multiplier* if, for each $a \in A$, $L_a b \in A$.

The set $M(A)$ of multipliers of the JB-algebra A is a unital JB-algebra and is the largest JB-subalgebra of A^{**} of A in which A is a Jordan ideal [10].

LEMMA 4.1. The JB-algebra A possesses an approximate identity.

PROPOSITION 4.2. If B is a JB-subalgebra of JB-algebra A containing an approximate identity for A, and operator commute, then $M(B) \subset M(A)$.

PROOF. Let $\{u_j\}$ be approximate identities for A contained in B. For $a \in A$ and $b \in M(B)$, $a \circ b = (\lim a \circ u_j) \circ b = a \circ (\lim u_j \circ b) \in A$ since $u_j \circ b \in B$ and operator commute. Hence $b \in M(A)$. Thus $M(B) \subset M(A)$.

For a JB-algebra A, A^+, the set of squares of elements

of A, is a positive cone which generates A. A JB-subalgebra B of A is said to be an *hereditary JB-subalgebra* if whenever $0 \leq a \leq b$ with $a \in A$ and $b \in B$ then $a \in B$.

LEMMA 4.3 [5]. Let A be a JB-algebra and J be an hereditary JB-subalgebra of A. Then

(i) The abelian elements of A form an hereditary and norm closed set.

(ii) Each abelian element of J is an abelian element of A.

LEMMA 4.4. Every non-zero closed quadratic ideal in the multiplier algebra $M(A)$ of the JB-algebra A has non-zero intersection with A.

PROOF. Let J be a non-zero closed quadratic ideal in $M(A)$ and let b be a non-zero element of the positive cone J^+ in J. It follows from [8] that $b^{1/2}$ is also an element of J^+.

For each element $a \in A$,

$$U_{b^{1/2}}a = 2(L_{b^{1/2}})^2 a - L_b a$$

is an element of A since both b and $b^{1/2}$ are elements of $M(A)$. Let $\{u_j\}$ be approximate identities for A. Then $\{U_{b^{1/2}} u_j\}$ is a bounded increasing net in A which possesses a least upper bound in A^{**}. It follows from [14, Lemma 2.2] that this least upper bound is b. Therefore, for some j,

$$U_{b^{1/2}}u_j \neq 0 \text{ and } 0 \leq U_{b^{1/2}}u_j \leq b.$$

Hence the positive cone J^+ in J is a closed face of the cone $M(A)^+$ and it follows that $U_{b^{1/2}}u_j$ is an element of $J \cap A$.

The following theorem is a Jordan Banach algebra version of C*-algebra case [2, Proposition 2.3].

THEOREM 4.5. Each non-zero hereditary JB-subalgebra of $M(A)$ has a non-zero intersection with A.

PROOF. By Lemma 4.4 and by the fact that norm-closed quadratic ideals of JB-algebra A are precisely the hereditary JB-subalgebras of A.

A Jordan ideal J in a JB-algebra A is said to be *essential* in A if every non-zero closed Jordan ideals in B has non-zero intersection with J.

THEOREM 4.6 [10]. (i) The JB-algebra A is essential Jordan ideal in its multiplier algebra $M(A)$. (ii) If the JB-algebra A is an essential Jordan ideal in a JB-algebra B then there exists a Jordan isomorphism from B into $M(A)$ which is the identity mapping on A.

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ON FINITELY GENERATED SEMIPRIME ALGEBRA OVER COMMUTATIVE RINGS

YOUNG IN KWON AND CHANG WOO HAN

1. Introduction

Let R be a commutative ring. E. P. Armendariz studied in [2] that a semiprime finitely generated R-algebra when R is a regular ring and that by combining the above fact with the results of [6, 7], R is semiprime and every $f.g.$ semiprime R-algebra A is Azumaya if the ring R is regular.

In this paper, we prove converse of Armendariz's theorem and we get a necessary and sufficient condition on which a regular ring R is π-regular.

That is, we have the following results;

1) Let R be a commutative ring. Then the following are equivalent;

i) R is von Neumann regular.

ii) R is semiprime and every $f.g.$ semiprime R-algebra is Azumaya.

2) Let R be a commutative ring. Then the following are equivalent;

i) R is von Neumann regular.

ii) Every integral extension of R is π-regular.

An algebra A is called *Azumaya* if R is both central and

separable. The ring R is said to be *P.I. ring* if R satisfies a polynomial identity with coefficients in the centroid and at least one coefficient is invertible. All other notations and terminologies will follow from [2] and [4].

2. Preliminaries

Kaplansky made the following conjecture in [4]: A ring R is von Neumann regular if and only if R is semiprime and each prime factor ring of R is von Neumann regular. That the conjecture fails to hold in general was shown by a counter example of J.W. Fisher and R.L. Snider.

THEOREM 2.1 [4]. A ring R is von Neumann regular if and only if R is semiprime, the union of any chain of semiprime ideals of R is a semiprime ideal of R and each prime factor ring of R is von Neumann regular.

Since any finitely generated algebra over a commutative ring satisfies a polynomial identity (is a P.I.-algebra), this leads to consideration of semiprime P.I.-algebra with regular center.

THEOREM 2.2 [2]. Let A be a semiprime finitely generated algebra over a commutative regular ring R. Then A is a regular ring.

The ring R is finitely generated as a ring over its center $Z(R)$, if R is an epimorphic image of a free (non commutative) ring over $Z(R)$ generated by finitely many indeterminates $[x_1, x_2, \cdots, x_n]$ which only commute with elements of $Z(R)$. Following C. Proces, the ring R is called an *affine ring* if R is finitely generated over its center $Z(R)$.

THEOREM 2.3 [7]. Let R be an affine ring. Then the following properties are equivalent;

1) Every simple right R-module is injective.
2) R is von Neumann regular.
3) R is biregular.

THEOREM 2.4[2]. Let A be an algebra over a regular ring with center of A being R. A is Azumaya over R if and only if A is a biregular ring which is finitely generated over R.

Combining Theorems 2.2, 2.3 and 2.4, we have the following result.

THEOREM 2.5 [2]. Let A be a finitely algebra over a regular ring. The following conditions on A are equivalent:

1) A is semiprime.
2) A is regular.
3) A is biregular.
4) A is semiprime Azumaya algebra.

The follwing theorem was shown by Storrer.

THEOREM 2.6 [4]. Let R be a P.I. ring. Then the following are equivalent:

1) R is π-regular.
2) Each prime ideal of R is primitive.
3) Each prime ideal of R is maximal.
4) R is left (right) π-regular.
5) $R/rad(R)$ is π-regular, where $rad(R)$ is prime radical.
6) Each prime factor ring of R is von Neumann regular.

3. Main results

LEMMA 3.1. Let R be a commutative prime ring and $0\neq a$

$\in R$. If $A=\begin{pmatrix} R & aR \\ aR & R \end{pmatrix}$ is Azumaya, then a is invertible in R.

PROOF. It is easily checked that R coincides with the center $Z(A)$. Now if A is Azumaya, $A\otimes_R A^{op}\cong Hom_R(A, A)$. In this case $\sigma(a\otimes b)(x)=axb$ for $x\in A$.

Consider $f\in Hom_R(A, A)$ such that $f(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix})=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Then since A is Azumaya, there are $\begin{pmatrix} x_i & ay_i \\ az_i & w_i \end{pmatrix}$ and $\begin{pmatrix} x_i' & ay_i' \\ az_i' & w_i' \end{pmatrix}$ in A, $1\leqq i\leqq n$ for some n such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}=\sum_{i=1}^{n}\begin{pmatrix} x_i & ay_i \\ az_i & w_i \end{pmatrix}\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_i' & ay_i' \\ az_i' & w_i' \end{pmatrix}.$$

By this relation, we have $1\in a^2R$ and so a is invertible in R.

THEOREM 3.2. Let R be a commutative ring. Then the following are equivalent.

1) R is von Neumann regular.

2) R is semiprime and every $f.g.$ semiprime R-algebra A is Azumaya.

PROOF. Assume that R is von Neumann regular. By Theorem 2.5, A is Azumaya algebra.

For the opposite direction, let P be a prime ideal of R. We will show that P is a maximal ideal of R. Now, take $a\in R$ and consider an R-algebra $A=\begin{pmatrix} R & aR \\ aR & R \end{pmatrix}$. Then A is a finitely generated semiprime algebra over R. In this case, the center $Z(A)=\left\{\begin{pmatrix} x & 0 \\ 0 & w \end{pmatrix} \mid (x-w)a=0\right\}$. By our assumption, A is separable over $Z(A)$.

Consider the mapping $\sigma: A \longrightarrow \begin{pmatrix} R/P & aR/P \\ aR/P & R/P \end{pmatrix}$ with $\sigma\left[\begin{pmatrix} x & ay \\ az & w \end{pmatrix}\right]$ $= \begin{pmatrix} x+p & ay+p \\ az+p & w+p \end{pmatrix}$, where $\bar{a}=a+P$. Then since $a \notin P$, we have that $Ker\ \sigma = \left\{\begin{pmatrix} x & ay \\ az & w \end{pmatrix} \middle| x,\ y,\ z,\ w \in P\right\} = PA$. Therefore $A/PA \cong \begin{pmatrix} R/P & \bar{a}R/P \\ \bar{a}R/P & R/P \end{pmatrix}$.

Now since $PZ(A)A = PA$ and A is Azumaya, we have $PA \cap Z(A) = PZ(A)$. So A/PA is Azumaya over $Z(A)/PZ(A)$. Also in this case $Z(A/PA) = Z(A)/PZ(A)$ [1]. But since $A/PA \cong \begin{pmatrix} R/P & \bar{a}R/P \\ \bar{a}R/P & R/P \end{pmatrix}$, we have $Z(A/PA) \cong R/P$. So $\begin{pmatrix} R/P & \bar{a}R/P \\ \bar{a}R/P & R/P \end{pmatrix}$ is Azumaya over R/P. Therefore, by our Lemma 3.1, $\bar{a}$ is invertible in R/P. Hence R/P is a field. Thus R is a von Neumann regular ring.

Corollary 3.3. Let R be a commutative ring, then the following are equivalent:

1) R is von Neumann regular.

2) R is semiprime and for every finitely generated R-algebra A, $J(A)$ is nilpotent and $A/J(A)$ is Azumaya.

Proof. In [2], E. P. Armendariz proved that if R is von Neumann regular then $J(A)$ is nilpotent and $A/J(A)$ is a regular ring.

Conversely, let P be a prime ideal and $a \notin P$. Then $A = \begin{pmatrix} R & aR \\ aR & R \end{pmatrix}$ is finitely generated semiprime R-regular. But since A is a normalizing finite extension of R, we have

$0=J(R)=R\cap J(A)$ and so $R\cong A/J(A)$. This shows that $A/J(A)$ is R-algebra.

Now since A is semiprime and $J(A)$ is nilpotent, $J(A)=0$. Therefore A is Azumaya. By Theorem 3.2, R is von Neumann regular.

Let A be a ring with identity. Consider the condition (*) the ring A satisfies a polynomial identity $f(x_1, x_2,\cdots,x_n)=0$ for which f has coefficient in C, the center of A, and for which at each prime ideal P of A, f induces a nontrivial polynomial identity on A/P.

THEOREM 3.4 [5]. Let A be a ring with identity which is integral over unital subring B of C, the center of A, suppose futher that B satisfies (*), then; If P is prime ideal of A, P is maximal ideal of A if and only if $P\cap A$ is maximal ideal of B.

THEOREM 3.5. Let R be a commutative ring. Then the following are equivalent;

1) R is von Neumann regular.
2) Every integral extension of R is π-regular.

PROOF. Suppose that R is von Neumann regular and A is integral extension of R. To show that A is π-regular, let P be a prime ideal of A. Then A/P is integral over $R/P\cap R$. Since P is a maximal ideal of A, $P\cap R$ is maximal ideal of R. Therefore $R/P\cap R$ is a field. By Theorem 2.6, A/P is π-regular. Thus A is π-regular. Conversely, since $A=\begin{pmatrix} R & R \\ R & R \end{pmatrix}$ is integral extension of R, it is π-regular. It follows that R is von Neumann regular.

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CERTAIN SUBCLASSES OF ANALYTIC P-VALENT FUNCTIONS

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1. Introduction

Let A_p denote the class of functions of the form

$$(1.1)\qquad f(z)=z^p+\sum_{k=1}^{\infty} a_{p+k}\, z^{p+k} \quad (p\in N=\{1,2,\cdots\})$$

which are analytic in the unit disk $U=\{z:|z|<1\}$. For $f(z)$ and $g(z)$ being in the class A_p, $f(z)$ is said to be subordinate to $g(z)$ if there exists a Schwarz function $w(z)$, $w(0)=0$ and $|w(z)|<1$ for $z\in U$, such that $f(z)=g(w(z))$. We denote by $f(z)\prec g(z)$ this relation. In particular, if $g(z)$ is univalent in the unit disk U, then the subordination $f(z)\prec g(z)$ is equivalent to $f(0)=g(0)$ and $f(U)\subset g(U)$.

A function $f(z)$ belonging to A_p is said to be in the class $S^*_p[a,b]$ if it satisfies

$$(1.2)\qquad \frac{zf'(z)}{pf(z)}\prec\frac{1+az}{1+bz}$$

for some a and b with $-1\leqq b<a\leqq 1$, and for all $z\in U$.

Further, a function $f(z)$ belonging to A_p is said to be in the class $K_p[a,b]$ if it satisfies $zf'(z)/p\in S^*_p[a,b]$.

The class $S^*_1[a,b]$ was introduced by Goel and Mehrok ([1], [2]), and Janowski [3]. Further, the class $K_1[a,b]$

was introduced by Silverman and Silvia [5].

Let $S^*_p(a,b)$ be the subclass of A_p consisting of functions which satisfy the condition

$$\left|\frac{zf'(z)}{pf(z)}-a\right|<b \tag{1.3}$$

for some a and b with $a\geqq b$, and for all $z\in U$. Furthermore we denote by $K_p(a,b)$ the subclass of A_p consisting of functions which satisfy the condition $zf'(z)/p \in S^*_p(a,b)$.

The classes $S^*_1(a,b)$ and $K_1(a,b)$ were introduced by Silverman [4] and Silverman and Silvia [5], respectively.

2. Some Properties

We begin with the statement and the proof of the following result.

THEOREM 1. If $-1<b<a\leqq 1$, then

$$S^*_p[a,b]\equiv S^*_p\left(\frac{1-ab}{1-b^2},\ \frac{a-b}{1-b^2}\right). \tag{2.1}$$

Further, if $a\geqq b$, then

$$S^*_p(a,b)\equiv S^*_p\left[\frac{b^2-a^2+a}{b},\ \frac{1-a}{b}\right]. \tag{2.2}$$

PROOF. We employ the same manner by Silverman and Silvia [5]. Let $f(z)\in S^*_p[a,b]$ with $-1<b<a\leqq 1$, that is,

$$\frac{zf'(z)}{pf(z)}\prec F(z)=\frac{1+az}{1+bz}. \tag{2.3}$$

By using the result due to Singh and Goel [6], we have

$$\left|F(z)-\frac{1-ab|z|^2}{1-b^2|z|^2}\right|\leqq\frac{(a-b)|z|}{1-b^2|z|^2}\quad (z\in U). \tag{2.4}$$

It follows from (2.4) that $F(z)$ maps the circle $|z|=1$ onto

a circle

$$(2.5)\qquad \left|w-\frac{1-ab}{1-b^2}\right|=\frac{a-b}{1-b^2}.$$

This implies that

$$(2.6)\qquad \left|\frac{zf'(z)}{pf(z)}-\frac{1-ab}{1-b^2}\right|<\frac{a-b}{1-b^2}\,(-1<b<a\leqq 1;\ z\in U).$$

Noting $(1-ab)/(1-b^2)\geqq(a-b)/(1-b^2)$ for $-1<b<a\leqq 1$, we obtain (2.1).

Next, let $f(z)\in S^*_p(a,b)$ with $a\geqq b$, that is,

$$(2.7)\qquad \left|\frac{zf'(z)}{pf(z)}-a\right|<b.$$

Then we have to find A and B with $-1<B<A\leqq 1$ such that $a=(1-AB)/(1-B^2)$ and $b=(A-B)/(1-B^2)$ for $a\geqq b$. Because we find such A and B, then we have

$$(2.8)\qquad \left|\frac{zf'(z)}{pf(z)}-\frac{1-AB|z|^2}{1-B^2|z|^2}\right|\leqq\frac{(A-B)|z|}{1-B^2|z|^2}$$

which implies

$$(2.9)\qquad \frac{zf'(z)}{pf(z)}<\frac{1+Az}{1+Bz},$$

or $f(z)\in S^*_p[A,B]$.

Letting $A=B+b(1-B^2)$ for $-1<B<A\leqq 1$, we obtain

$$(2.10)\qquad bB^3+(a-1)B^2-bB+1-a=0.$$

The above equation (2.10) has the solutions $B=\pm 1$, $(1-b)/a$. Since $-1<B<A\leqq 1$, we only take $B=(1-b)/a$. Thus we have $B=(1-b)/a$ and $A=(b^2-a^2+a)/b$. This completes the proof of (2.2).

THEOREM 2. If $-1<b<a\leqq 1$, then

$$(2.11) \qquad K_p[a, b] \equiv K_p\left(\frac{1-ab}{1-b^2}, \ \frac{a-b}{1-b^2}\right).$$

Further, if $a \geqq b$, then

$$(2.12) \qquad K_p(a, b) \equiv K_p\left[\frac{b^2-a^2+a}{b}, \ \frac{1-a}{b}\right].$$

PROOF. Note that $f(z) \in K_p[a, b]$ if and only if $zf'(z)/p \in S^*_p[a, b]$, and that $f(z) \in K_p(a, b)$ if and only if $zf'(z)/p \in S^*_p(a, b)$. Therefore the proof of Theorem 2 follows form Theorem 1.

Next we prove

THEOREM 3. $S^*_p(a_1, b_1) \subseteq S^*_p(a_2, b_2)$ if and only if $|a_2-a_1| \leqq b_2-b_1$. Furthermore, $S^*_p[a_1, b_1] \subseteq S^*_p[a_2, b_2]$ if and only if $|a_2b_1-a_1b_2| \leqq (a_2-a_1)-(b_2-b_1)$.

PROOF. Since $S^*_p(a_1, b_1) \subseteq S^*_p(a_2, b_2)$ if and only if

$$\{w: |w-a_1| < b_1\} \subseteq \{w: |w-a_2| < b_2\},$$

or, if and only if $a_2-b_2 \leqq a_1-b_1$ and $a_1+b_1 \leqq a_2+b_2$, we have $S^*_p(a_1, b_1) \subseteq S^*_p(a_2, b_2)$ if and only if $|a_2-a_1| \leqq b_2-b_1$. In view of Theorem 1, we note that $S^*_p[a_1, b_1] \subseteq S^*_p[a_2, b_2]$ if and only if

$$\{w: |w-A_1| < B_1\} \subseteq \{w: |w-A_2| < B_2\},$$

where $A_j = (1-a_jb_j)/(1-b_j^2)$ and $B_j = (a_j-b_j)/(1-b_j^2)$, which equivalent to $|A_2-A_1| \leqq B_2-B_1$. Thus we have $S^*_p[a_1, b_1] \subseteq S^*_p[a_2, b_2]$ if and only if $|a_2b_1-a_1b_2| \leqq (a_2-a_1) -(b_2-b_1)$.

Finally, we have

THEOREM 4. $K_p(a_1, b_1) \subseteq K_p(a_2, b_2)$ if and only if $|a_2-a_1| \leqq b_2-b_1$. Furthermore, $K_p[a_1, b_1] \subseteq K_p[a_2, b_2]$ if and only if $|a_2b_1 -a_1b_2| \leqq (a_2-a_1)-(b_2-b_1)$.

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PRIME DUAL IDEALS IN TANAKA ALGEBRAS

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1. Introduction

K. Iseki [6] has introduced the notion of a BCK-algebra which is an algebraic formulation of a propositional calculus. We refer to Iseki [7], [8], and [9] for certain basic properties of these algebras. The ideals and their properties were studied by K. Iseki and S. Tanaka [10]. Elias Deeba [5] has introduced the notion of dual ideals in BCK-algebras. In [1], B. Ahmad has given a characterization of prime dual ideals in Tanaka algebras. In this note, we obtain some properties of prime dual ideals in Tanaka algebras. We recall that a set X is said to be a Tanka algebra [9] if the following conditions are satisfied:

(1) $(X, \leq)$ is a partially ordered set with least element 0,

(2) $(x*y)*z=(x*z)*y$,

(3) $x*(x*y)=y*(y*x)$,

where $x \leq y$ means $x*y=0$.

S. Tanaka proved that the algebra X is a semilattice with respect to $x \wedge y$ which is defined by $y*(y*x)$, and X is a BCK-algebra, i.e. $(x*y)*(x*z) \leq z*y$ holds in X (See S. Tanaka [11], [12]).

In a BCK-algebra, the notion of a dual ideal has been defined in [5] as follows:

DEFINITION. A non-empty subset D of X is a *dual ideal* in X if the following conditions are satisfied:

(1) $x \in D$, $x \leq y$ imply $y \in D$.

(2) $x \in D$, $y \in D$ imply there exists an element $z \in D$ such that $z \leq x$, $z \leq y$.

In [1], B. Ahmad has defined a prime dual ideal as follows:

DEFINITION. A dual ideal P in a Tanaka algebra X is called a *prime dual ideal* if for any x, y, $x \vee y \in P$ implies $x \in P$ or $y \in P$.

2. Main Results

LEMMA 1. Let $\{P_i\}_{i \in I}$ be a non-empty family of prime dual ideals in a bounded Tanaka algebra X. If the family is totally ordered by set inclusion, then both $\bigcup_i P_i$ and $\bigcap_i P_i$ are prime dual ideals.

PROOF. To prove $\bigcup_i P_i$ (put P') is a prime dual ideal, suppose $x \in P'$ and $x \leq y$ for every $x, y \in X$. Then we have $x \in P_i$ and $x \leq y$ for some $i \in I$, which imply $y \in P_i$ for some i. This means that $y \in P'$. Assume that x and y are in P'. Then $x \in P_i$ and $y \in P_j$ for some i, j. If $P_i \subset P_j$ then $x, y \in P_j$, and hence there exists an element $z \in P_j \subset P'$ such that $z \leq x$ and $z \leq y$. If $P_j \subset P_i$ then $x, y \in P_j$, and therefore there exists $z \in P_i \subset P'$ such that $z \leq x$ and $z \leq y$. Thus in any case there exists an element $z \in P'$ with $z \leq x$ and $z \leq y$. It follows that P' is a dual ideal. Next suppose that $x \vee y \in P'$ and $x \notin P'$. Then $x \vee y \in P_i$ and $x \notin P_i$ for some i, which imply $y \in P_i \subset P'$. Therefore P' is a prime dual ideal.

To prove $\bigcap_i P_i$ (put P'') is a prime dual ideal, we first

assume that $x\in P''$ and $x\leq y$ for all $x, y\in X$. Then $x\in P_i$ and $x\leq y$ for all $i\in I$. This imply that $y\in P_i$ for all i, and hence $y\in P''$. If x and y are in P'' then $x, y\in P_i$ for all i. Then there exists $z\in P_i$ such that $z\leq x$ and $z\leq y$ for all i. It follows that $z\in P''$ with $z\leq x$ and $z\leq y$. Thus P'' is a dual ideal of X. Now suppose that $x\vee y$ belongs to P'' but $x\notin P''$. Then it is possible to choose i with $x\vee y\in P_i$ but $x\notin P_i$. Then $y\in P_i$. Finally let j be an arbitrary element of I. If $P_i\subset P_j$ then $y\in P_j$. On the other hand, if $P_j\subset P_i$ then $x\vee y\in P_j$ while $x\notin P_j$. Consequently $y\in P_j$. Thus in any case $y\in P_j$ and hence $y\in P''$. Therefore P'' is a prime dual ideal and the proof is complete.

PROPOSITION 2. Let D be a dual ideal of a bounded Tanaka algebra X and let P be a prime dual ideal containing D. Then P contains a prime dual ideal which contains D and has no smaller prime dual ideal containing D.

PROOF. Denote by $\mathscr{P}$ the set of all prime dual ideals which contain D and are contained in P. Then $\mathscr{P}$ is not empty. Define a relation $\leq$ on $\mathscr{P}$ by $P'\leq P''$ if and only if $P''\subset P'$ for all P', $P''\in\mathscr{P}$. Then $(\mathscr{P}, \leq)$ is a partially ordered set. Let S be a non-empty totally ordered subset of $\mathscr{P}$. By the above Lemma, the intersection of all members of S is a prime dual ideal $\bar{P}$, say. This certainly contains D and is contained in P. Consequently $\bar{P}\in\mathscr{P}$. Since $\bar{P}\subset P'$ for all $P'\in S$, we have $P'\leq\bar{P}$ for every $P'\in S$. Thus $\bar{P}$ is an upper bound for S. By Zorn's Lemma, $\mathscr{P}$ contains a maximal element P^*, and hence P^* is a prime dual ideal and $D\subset P^*\subset P$. Suppose now that P^{**} is a prime dual ideal satisfying $D\subset P^{**}\subset P^*$. Then $P^{**}\in\mathscr{P}$ and $P^*\leq P^{**}$. By

the maximality of P^*, we have $P^* = P^{**}$, which completes the proof.

LEMMA 3. Let X be a bounded and implicative BCK-algebra and D be a dual ideal of X. Then D is maximal dual implies D is a prime dual ideal.

PROOF. See [3], p. 650.

PROPOSITION 4. Let X be a bounded and implicative Tanaka algebra, A an ideal of X, and let D be a dual ideal of X such that $D \cap A = \phi$. Then X contains a prime dual ideal which contains D and disjoint from A.

PROOF. Let $\mathscr{D}$ be the set of all dual ideals which contain D and disjoint from A. $\mathscr{D}$ is non-empty because $D \in \mathscr{D}$. We shall show that $\mathscr{D}$ is inductively ordered by inclusion. To this purpose, let $\mathscr{D}'$ be a totally ordered non-empty subset of $\mathscr{D}$. Let E be the union of all dual ideals in $\mathscr{D}'$. Then, by Lemma 1, E is a dual ideal. Also E contains D and disjoint from A, which implies that $E \in \mathscr{D}$. Moreover, it is clear that E is an upper bound of $\mathscr{D}'$. By Zorn's Lemma, $\mathscr{D}$ has a maximal element, say P. It follows from Lemma 3 that P is a prime dual ideal.

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BAYESIAN SHRINKAGE ESTIMATION OF THE RELIABILITY FUNCTION FOR THE LEFT TRUNCATED EXPONENTIAL DISTRIBUTION

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1. Introduction.

Consider the two parameters exponential distribution with positivity constraint on the truncation parameter defined by the probability density function,

(1) $f(x|\theta, \lambda)=\lambda^{-1}\exp[-\lambda^{-1}(x-\theta)],\ x>\theta,\ \lambda>0,$

where $\theta>0$ for the density to be left truncated.
The model (1) will be referred to as the left truncated exponential distribution.

It is well-known that the left truncated exponential distribution is really appropriate as a lifetime distribution model for reliability and life-testing.

Evans and Nigm (1980) investigated that the use of the two parameters exponential distribution with no positivity constraint on the truncation parameter as a lifetime distribution model is unrealistic and may lead to inefficient inferences and prediction.

Both classical and Bayesian estimation of the reliability function for the two parameters exponential distribution with or with no positivity constraint on the truncation

parameter have studied by many authors, including Basu (1964), Varde(1969), Grubbs(1971), Pierce(1973), Sinha and Guttman(1976), Sinha and Kale(1980), Martz and Waller (1982), Trader(1985) and so on.

Trader(1985) considered the truncated normal distribution as a prior distribution for the truncation parameter in the left truncated exponential distribution.

The shrinkage estimation techniques have been advocated by a number of authors as a procedure for lowering the mean squared-error (MSE) of the minimum variance unbiased estimator (MVUE) or maximum likelihood estimator(MLE) [see Thompson(1968), Mehta and Srinivasan(1971), Pandey and Singh(1980), Pandey and Srivastava(1985) and so on].

In recent year, the use of Bayes shrinkage estimation of the parameters for binomial, Poisson and normal distributions were considered by Lemmer (1981) at first. But he has been little paid attention to lifetime distributions-exponential, Weibull etc.

Pandey and Upadhyay(1985) considered the Bayes shrinkage estimation of the parameters for Weibull distribution, and discussed the relative s-efficiencies of these Bayes shrinkage estimators with respect to the unbiased estimators of Engelhardt and Bain(1977) on the basis of a Monte Carlo study of 500 random samples.

In this study, we will consider some Bayes shrinkage estimators of the reliability function in the left truncated exponential distribution.

First, we will give the MVUE and Bayes estimators of the reliability function with the noninformative and conjugate prior distributions in this model.

Next, using the Bayes estimator instead of the guessed value which is close to the true unknown value, such as that given by Pandey(1985), we will propose some Bayes shrinkage estimators of the reliability function in this model.

Finally, we will compare the relative s-efficiencies of the Bayes shrinkage estimators with respect to the MVUE by the Monte Carlo simulation and numerical evaluation technique in the sense of MSE.

2. MVUE and Bayes estimators of the reliability function.

Let $x_{(1)} \leqq x_{(2)} \leqq \cdots \leqq x_{(r)}$ be the first r ordered observations of n failure times form the left truncated exponential distibution(1) under test without replacement.

For a given time t, the reliability function, the probability that survival until time t, is given by

$$(2) \quad R_t = 1 - F(t) = \int_t^{\infty} f(x)\,dx$$
$$= \exp\left(-\frac{t-\theta}{\lambda}\right), \quad t \geqq \theta,$$

where F is the cummulative distribution function of the failure time x.

Basu(1964) obtained the MVUE of the reliability function at time t to be

$$(3) \quad \hat{R}_t = \begin{cases} \dfrac{n-1}{n}\left(1 - \dfrac{t - x_{(1)}}{y_r}\right)^{r-2}, & x_{(1)} < t < x_{(1)} + y_r \\ 1, & x_{(1)} > t \\ 0, & t > x_{(1)} + y_r, \end{cases}$$

where $y_r = \sum_{i=1}^{r} (x_{(i)} - x_{(1)}) + (n-r)(x_{(r)} - x_{(1)})$.

The likelihood function is given by

$$(4)\quad L(\theta,\lambda|x_{(1)},\ x_{(2)},\cdots,\ x_{(r)})_{\infty}\lambda^{-r}\ \exp\{-\lambda^{-1}[y_r+n(x_{(1)}-\theta)]\},\quad 0<\theta<x_{(1)},\ \lambda>0.$$

The noninformative joint prior distribution of θ and λ is taken as [Sinha and Guttman(1976)]

$$(5)\quad g_1(\theta,\lambda)=\begin{cases}(1-b) & ,\quad \text{if } \theta=\theta_0,\ \lambda=\lambda_0\\ b/\lambda^a & ,\quad \text{otherwise,}\end{cases}$$

where $a>0$, $0\leq b\leq 1$, $\lambda>0$, $0<\theta<x_{(1)}$, θ_0, λ_0 are the prior values in the vicinities of the true values θ and λ, respectively, and the prior distribution has weight $(1-b)$ in the prior values and weight b in the rest intervals.

We obtain the joint posterior distribution of θ and λ as

$$(6)\quad g_1(\theta,\lambda|x_{(1)},x_{(2)},\cdots,x_{(r)})$$
$$=\frac{L(\theta,\lambda|x_{(1)},\ x_{(2)},\cdots,\ x_{(r)})g_1(\theta,\ \lambda)}{\int_0^{x(1)}\int_0^{\infty}L(\theta,\ \lambda|x_{(1)},\ x_{(2)},\cdots,\ x_{(r)})g_1(\theta,\lambda)\ d\lambda d\theta}.$$

Therefore, from (4), (5) and (6), the Bayes estimator of the reliability function with the noninformative prior distribution under the squared-error loss can be written as

$$(7)\quad R^*_{t1}=E[R_t|x_{(1)},\ x_{(2)},\cdots,\ x_{(r)}]$$
$$=\int_0^{x(1)}\int_0^{\infty}R_t\,g_1(\theta,\lambda|x_{(1)},\ x_{(2)},\cdots,\ x_{(r)})\ d\lambda d\theta$$
$$=\frac{P_{r+a-2}(p_1+1,\ p_2+t-x_{(1)},\ p_3+t)+Q(q_1+t-\theta_0)}{P_{r+a-2}(p_1,\ p_{2,},\ p_3)+Q(q_1)}$$

where

$$P_{r+a-2}(p_1,p_2,p_3)=b\ \frac{\Gamma(r+a-2)}{p_1}\ \frac{1}{p_2{}^{r+a-2}}\ \left(1-\left(\frac{p_2}{p_3}\right)^{r+a-2}\right),$$

$$Q(q_1)=(1-b)\lambda_0^{-r}\ \exp\left(-\frac{q_1}{\lambda_0}\right),$$

$$p_1=n,\ p_2=y_r,\ p_3=y_r+nx_{(1)},\ q_1=y_r+nx_{(1)}-n\theta_0.$$

Also, we can use the conjugate joint prior distribution of θ and λ as [Evans and Nigm(1980)]

$$(8)\quad g_2(\theta,\ \lambda)=\begin{cases}(1-b)\ , & \text{if } \theta=\theta_0\ ,\ \lambda=\lambda_0\\ b\dfrac{1}{\lambda^c}\exp\left\{-\dfrac{d+h(m-\theta)}{\lambda}\right\}, & \text{otherwise,}\end{cases}$$

where to be a proper prior distribution we must have $c>2$, $d>0$, $h>0$, and $0<\theta<m$, $\lambda>0$.

From (8), the joint posterior distribution of θ and λ becomes

$$(9)\quad g_2(\theta,\ \lambda|x_{(1)},\ x_{(2)},\cdots,\ x_{(r)})$$
$$=\frac{L(\theta,\ \lambda|x_{(1)},\ x_{(2)},\cdots,\ x_{(r)})\,g_2(\theta,\lambda)}{\int_0^A\int_0^\infty L(\theta,\ \lambda|x_{(1)},\ x_{(2)},\cdots,x_{(r)})g_2(\theta,\ \lambda)d\lambda d\theta},$$

where $A=\min(m,\ x_{(1)})$.

Therefore, from (4), (8) and (9), the Bayes estimator of the reliability function with the conjugate prior distribution under the squared-error loss can be written as

$$(10)\quad R^*_{t2}=E(R_t|x_{(1)},x_{(2)},\cdots,x_{(r)})$$
$$=\int_0^A\int_0^\infty R_t\,g_2(\theta,\ \lambda|x_{(1)},\ x_{(2)},\cdots,\ x_{(r)})d\lambda d\theta$$
$$=\frac{P_{r+c-2}(\,p_1+h+1,\ p_2+nx_{(1)}+d+hm+t-(n+h+1)A,\ \ p_3+d+hm+t)+Q(q_1+t-\theta_0))}{P_{r+c-2}(p_1+h,\ \ p_2+nx_{(1)}+d+hm-(n+h)A,\ p_3+d+hm)+Q(q_1)}$$

3. Bayes shrinkage estimators of the reliability function.

Let us consider a shrunken estimator of the form

$$T=k(\hat{\theta}-\theta_0)+\theta_0,$$

where $0\leq k\leq 1$, $\hat{\theta}$ is the MVUE of θ and θ_0 is a prior value which is close to the true unknown value θ.

This shrinkage estimator for θ was considered by Thompson (1968) at first, and showed that T is more efficient than MVUE for mean parameter in the sense of MSE when sample size is small and θ_0 is in the vicinity of true value θ in the normal, Poisson, binomial and gamma population.

Now we propose two classes of the Bayes shrinkage estimator of the reliability function:

$$(12)\quad T_{Rt1}=k_1(\hat{R}_t-R^*_{t1})+R^*_{t1} =k_1\hat{R}_t+(1-k_1)R^*_{t1},$$

$$(13)\quad T_{Rt2}=k_2(\hat{R}_t-R^*_{t2})+R^*_{t2} =k_2\hat{R}_t+(1-k_2)R^*_{t2},$$

where $0\leq k_1, k_2\leq 1$, $\hat{R}_t$ is the MVUE of R_t, $R^*_{t_1}$ is the Bayes estimator of R_t with the noninformative prior distribution and R^*_{t2} is the Bayes estimator of R_t with the conjugate prior distribution.

4. Comparisons of the relative s-efficiencies of the Bayes shrinkage estimators with respect to the MVUE.

The relative s-efficiencies of the Bayes shrinkage estimators with respect to the MVUE of the reliability function are given by

$$(14)\quad \mathrm{REF}_1(T_{Rt1}, \hat{R}_t)=\mathrm{Var}(\hat{R}_t)/\mathrm{MSE}(T_{Rt1}),$$

$$(15)\quad \mathrm{REF}_2(T_{Rt2}, \hat{R}_t)=\mathrm{Var}(\hat{R}_t)/\mathrm{MSE}(T_{Rt2}),$$

where $\hat{R}_t$ is the MVUE of R_t, T_{Rt1} is the Bayes shrinkage

estimator of R_t with the noninformative prior distribution (5) and T_{Rt2} is the Bayes shrinkage estimator with the conjugate prior distribution (8).

A Monte Carlo study has been performed on the relative s-efficiencies of the proposed Bayes shrinkage estimators with respect to the MVUE and the numerical values of the relative s-efficiencies of proposed Bayes shrinkage estimators with respect to the MVUE have been evaluated by use of the computer system.

The Monte Carlo simulation on the relative s-efficiencies of T_{Rt1} and T_{Rt2} with respect to $\hat{R}_t$ has been performed as the following four parts.

Part 1: 500 random samples of the first r ordered failure times were generated from the left truncated expontial distribution (1) with the parameters λ and θ such that θ/θ_0 is fixed at 1 and λ/λ_0: 0.50(0.25) 1.75 varies with (n, r), and REF_1 $(T_{Rt1}, \hat{R}_t)$ were evaluated for $a=1, b=0.2$ (0.2) 0.8 and $k_1=0.2$ (0.2) 0.8 to avoid complexity on the table 1.

Part 2: 500 random samples of the first r ordered times were generated from the left truncated exponential distribution (1) with the parameters λ and θ such that λ/λ_0 is fixed at 1 and θ/θ_0: 0.50(0.25) 1.75 varies with (n, r), and $\mathrm{REF}_1(T_{Rt1}, \hat{R}_t)$ were evaluated for $a=2$, $b=0.2(0.2)$ 0.8 and $k_1=0.2(0.2)$ 0.8 on the table 2.

Part 3: 500 radom samples of the first r ordered failure times were generated from the left truncated exponential distribution (1) with the parameters λ

and θ such that θ/θ_0 is fixed at1 and λ/λ_0: 0.50 (0.25) 1.75 varies with (n, r), and REF_2 $(T_{Rt2}, \hat{R}_t)$ were evaluated for $c=4$, $d=2$, $h=1$, $m=2$, $b=0.2(0.2)$ 0.8 and $k_2=0.2(0.2)$ 0.8 on the table 3.

Part 4: 500 random samples of the first r ordered failure times were generated from the left truncated exponential distribution (1) with the parameters λ and θ such that λ/λ_0 is fixed at 1 and θ/θ_0: 0.50(0.25) 1.75 varies with (n, r), and $\mathrm{REF}_2(T_{Rt2}, \hat{R}_t)$ were evaluated for $c=5$, $d=2$, $h=1$, $m=2$, $b=0.2(0.2)$ 0.8 and $k_2=0.2(0.2)$ 0.8 on the table 4.

Throughout the table 1-4, we obtain the following results:

(a) T_{Rt1} is more efficient than MVUE $\hat{R}_t$ in the sense of MSE for all possible values of n, r, a, b and k_1 contained the effective interval which is in the vicinity of true value λ or θ.

(*b*) T_{Rt2} is also much more efficient than MVUE $\hat{R}_t$ in the sense of MSE for all possible values of n, r, c, d, h, m, b and k_2 contained the effective interval which is in the vicinity of true value λ or θ.

(c) When the guessed value λ_0 is true, that is λ/λ_0 is 1, T_{Rt1} and T_{Rt2} are most efficient in the sense of MSE.

(d) T_{Rt2} is more efficient than T_{Rt1} in the sense of MSE.

Table 1. *Relative s-efficiencies of* T_{Rt1} *with respect to* $\hat{R}_t$

(a=1)

λ/λ_0	n	r \ k_1, b	0.2				0.4				0.6				0.8			
			0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8
0.50	10	7	0.889	1.041	1.117	1.163	0.969	1.066	1.113	1.141	1.020	1.067	1.090	1.105	1.032	1.044	1.052	1.057
	20	15	0.785	0.950	1.036	1.039	0.856	0.979	1.040	1.077	0.920	0.998	1.035	1.057	0.970	1.005	1.021	1.031
0.75	10	7	1.566	1.385	1.292	1.234	1.477	1.316	1.239	1.192	1.333	1.221	1.168	1.136	1.166	1.112	1.087	1.071
	20	15	1.228	1.198	1.168	1.144	1.247	1.181	1.144	1.119	1.207	1.138	1.106	1.086	1.118	1.076	1.057	1.046
1.00	10	7	2.084	1.587	1.384	1.269	1.727	1.424	1.291	1.212	1.431	1.269	1.193	1.146	1.192	1.127	1.095	1.074
	20	15	2.166	1.588	1.351	1.216	1.767	1.417	1.260	1.166	1.448	1.261	1.170	1.113	1.197	1.123	1.083	1.057
1.25	10	7	1.659	1.426	1.313	1.244	1.461	1.309	1.236	1.192	1.286	1.198	1.157	1.132	1.132	1.095	1.077	1.067
	20	15	1.366	1.272	1.210	1.163	1.278	1.201	1.157	1.126	1.185	1.132	1.104	1.036	1.091	1.065	1.052	1.044
1.50	10	7	1.297	1.249	1.224	1.207	1.222	1.186	1.173	1.165	1.146	1.123	1.117	1.115	1.072	1.061	1.059	1.059
	20	15	0.988	1.011	1.075	1.102	0.999	1.034	1.060	1.032	1.005	1.025	1.042	1.058	1.005	1.013	1.022	1.030
1.75	10	7	1.126	1.152	1.170	1.184	1.093	1.118	1.135	1.143	1.068	1.031	1.093	1.105	1.035	1.041	1.048	1.054
	20	15	0.890	0.975	1.031	1.033	0.919	0.931	1.031	1.039	0.947	0.992	1.024	1.050	0.974	0.997	1.013	1.027

Table 2. Relative s-efficiencies of T_{Rt1} with respect to $\widehat{R}_t$

(a=2)

θ/θ_0	n	r \ k_1, b	0.2				0.4				0.6				0.8			
			0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8
0.50	10	7	2.055	2.087	2.109	2.113	2.530	2.483	2.406	2.254	2.220	2.155	2.067	1.917	1.538	1.508	1.474	1.414
	20	15	0.930	0.930	0.930	0.932	1.466	1.466	1.467	1.468	1.857	1.857	1.857	1.856	1.551	1.551	1.551	1.550
0.75	10	7	5.136	3.578	2.704	2.061	3.188	2.498	2.053	1.692	2.051	1.784	1.585	1.404	1.396	1.315	1.247	1.179
	20	15	8.086	6.790	5.326	3.603	4.795	4.143	3.436	2.576	2.612	2.411	2.170	1.833	1.546	1.498	1.434	1.334
1.00	10	7	2.887	2.146	1.815	1.614	2.138	1.740	1.544	1.419	1.620	1.428	1.325	1.256	1.259	1.188	1.147	1.118
	20	15	3.017	2.029	1.623	1.384	2.190	1.672	1.426	1.271	1.640	1.393	1.260	1.170	1.264	1.174	1.120	1.080
1.25	10	7	2.126	1.763	1.613	1.528	1.732	1.513	1.419	1.364	1.426	1.309	1.256	1.224	1.187	1.140	1.118	1.104
	20	15	1.540	1.305	1.279	1.241	1.375	1.248	1.199	1.173	1.231	1.157	1.127	1.111	1.108	1.074	1.061	1.053
1.50	10	7	1.849	1.632	1.548	1.502	1.568	1.432	1.377	1.348	1.339	1.263	1.232	1.215	1.153	1.121	1.107	1.100
	20	15	1.311	1.254	1.234	1.223	1.222	1.182	1.168	1.161	1.141	1.116	1.108	1.103	1.067	1.056	1.052	1.050
1.75	10	7	1.722	1.576	1.520	1.491	1.489	1.395	1.360	1.341	1.296	1.242	1.222	1.211	1.135	1.112	1.103	1.098
	20	15	1.254	1.231	1.223	1.219	1.183	1.167	1.161	1.158	1.117	1.107	1.103	1.102	1.056	1.051	1.050	1.049

Table 3. Relative s-efficiencies of T_{Rt2} with respect to $\widehat{R}_t$

(c=4)

λ/λ_0	n	r \ k_2, b	0.2: 0.2	0.2: 0.4	0.2: 0.6	0.2: 0.8	0.4: 0.2	0.4: 0.4	0.4: 0.6	0.4: 0.8	0.6: 0.2	0.6: 0.4	0.6: 0.6	0.6: 0.8	0.8: 0.2	0.8: 0.4	0.8: 0.6	0.8: 0.8
0.50	10	7	0.459	0.563	0.699	0.946	0.651	0.731	0.832	1.002	0.890	0.908	0.949	1.032	1.058	1.021	1.014	1.031
	20	15	0.272	0.358	0.472	0.670	0.391	0.478	0.585	0.753	0.576	0.416	0.721	0.839	0.821	0.836	0.869	0.924
0.75	10	7	2.234	2.258	2.252	2.201	2.321	2.146	1.998	1.943	1.933	1.765	1.633	1.509	1.414	1.343	1.285	1.227
	20	15	1.106	1.188	1.256	1.332	1.460	1.420	1.376	1.333	1.592	1.460	1.356	1.264	1.364	1.276	1.207	1.144
1.00	10	7	11.442	7.537	5.311	3.636	4.653	3.810	3.139	2.485	2.442	2.221	2.011	1.768	1.490	1.436	1.380	1.307
	20	15	13.637	8.848	5.803	3.529	4.989	4.091	3.270	2.423	2.517	2.293	2.048	1.738	1.506	1.454	1.390	1.297
1.25	10	7	3.723	3.362	3.001	2.581	2.941	2.639	2.363	2.061	2.058	1.911	1.772	1.613	1.417	1.369	1.322	1.263
	20	15	2.023	1.907	1.799	1.685	1.992	1.822	1.686	1.548	1.698	1.572	1.471	1.367	1.325	1.271	1.227	1.177
1.50	10	7	1.872	1.864	1.862	1.862	1.869	1.779	1.714	1.651	1.631	1.544	1.479	1.417	1.302	1.260	1.227	1.194
	20	15	0.909	0.956	1.017	1.110	1.053	1.062	1.084	1.126	1.134	1.112	1.106	1.110	1.112	1.089	1.075	1.066
1.75	10	7	1.365	1.421	1.486	1.582	1.431	1.420	1.426	1.453	1.366	1.326	1.307	1.301	1.202	1.173	1.156	1.146
	20	15	0.723	0.802	0.884	1.001	0.840	0.890	0.944	1.022	0.942	0.961	0.987	1.029	1.002	1.000	1.006	1.021

Table 4. Relative s-efficiencies of T_{Rt2} with respect to $\hat{R}_t$

(c=5)

θ/θ_0	k_2 →		0.2				0.4				0.6				0.8			
	n	r \ b	0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8
0.50	10	7	2.035	2.045	2.062	2.096	2.592	2.593	2.593	2.588	2.297	2.292	2.282	2.256	1.562	1.560	1.555	1.544
	20	15	0.930	0.930	0.930	0.930	1.466	1.466	1.466	1.466	1.857	1.857	1.857	1.857	1.552	1.552	1.552	1.552
0.75	10	7	12.821	10.956	8.918	6.507	5.435	4.946	4.389	3.639	2.681	2.560	2.413	2.192	1.550	1.523	1.489	1.433
	20	15	9.253	9.199	9.049	8.628	5.470	5.401	5.278	4 998	2.805	2.784	2.747	2.663	1.590	1.585	1.577	1.557
1.00	10	7	10.763	7.203	5.207	3.658	4.580	3.786	3.171	2.560	2.431	2.225	2.036	1.815	1.488	1.439	1.390	1.326
	20	15	14.309	9.671	6.540	4.035	5.127	4.325	3.550	2.632	2.553	2.363	2.147	1.852	1.515	1.472	1.418	1.335
1.25	10	7	6.326	4.469	3.442	2.651	3.624	2.962	2.503	2.099	2.195	1.981	1.808	1.631	1.435	1.378	1.327	1.270
	20	15	4.842	3.266	2.432	1.844	3.103	2.405	1.959	1.603	2.028	1.761	1.558	1.375	1.391	1.311	1.242	1.173
1.50	10	7	4.581	3.419	2.733	2.282	3.071	2.536	2.191	2.901	2.034	1.831	1.681	1.541	1.396	1.337	1.289	1.239
	20	15	2.643	1.991	1.678	1.475	2.116	1.715	1.504	1.361	1.648	1.443	1.324	1.239	1.278	1.202	1.153	1.117
1.75	10	7	3.759	2.921	2.459	2.113	2.743	2.297	2.022	1.802	1.924	1.736	1.606	1.463	1.367	1.309	1.264	1.222
	20	15	1.943	1.617	1.470	1.382	1.699	1.468	1.361	1.296	1.441	1.306	1.240	1.199	1.203	1.147	1.118	1.099

5. Conclusions.

In the comparisons of the Monte Carlo relative s-efficiencies of the proposed Bayes shrinkage estimators for the reliability function with respect to the MVUE in the left truncated exponential distribution based on type II censoring, the proposed estimators are more efficient than MVUE in the sense of MSE for all possible values of $n, r, a, b, c, d, h, m, k_1$ and k_2 if λ/λ_0 and/or θ/θ_0 approach 1. Also, the Bayes shrinkage estimator with the conjugate prior distribution is more efficient than the Bayes shrinkage estimator with the noninformative prior distribution.

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FIXED POINTS FOR FAMILY OF MAPPINGS*

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Common fixed point theorems for family of mappings on 2-metric spaces (Gähler 1963/64) have been established among others by Lal-Singh (1978), Rhoades (1979), Singh (1979), Ram (1982) & Cho (1985). Ram (1982) established the following result:

THEOREM 1. Let $\{S_n\}$ be a sequence of mappings from a complete 2-metric space X to itself. Let T be a continuous mapping from X to itself such that T and S_n commute and $S_n(X) \subseteq T(X)$, $n=1,2,3\cdots$. If there exists a positive number $q<1$ such that for every pair i, j, $i \neq j$,

$$(1.1) \qquad d(S_i x,\ S_j y,\ a) \leq q.\max\Big\{d(Tx, Ty, a),\ d(S_i x, Tx, a),\ d(S_j y, Ty, a),\ \frac{1}{2}[d(S_i x, Ty, a)+d(S_j y, Tx, a)]\Big\}$$

for all $x, y,\ a$ in X, then T and the sequence $\{S_n\}$ of mappings have a unique common fixed point. We prove the following:

* The intent of this paper is to offer common fixed point theorems for a countable family of mappings on 2-metric spaces.
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THEOREM 2. Let $\{S_n\}$ be a sequence of mappings from a 2-metric space X to itself. Let T be a mapping from X to itself such that

(2.1) $S_n(X)\subseteq T(X)$, $n=1,2,3\cdots$,

(2.2) $T(X)$ is a complete subspace of X.

If there exists a positive number $q<1$ such that for every pair i,j, $i\neq j$, the condition (1.1) holds, then T and $S_n(n=1,2,3,\cdots)$ have a coincidence point, i.e. there exists a point z such that

$$Tz=S_nz,\ n=1,2,3,\cdots.$$

Further, if T and S_n, $n=1,2,3,\cdots$, commute at z then T and S_n, $n=1,2,3\cdots$, have a unique common fixed point. Indeed Tz is the unique common fixed point.

PROOF. Pick x_0 in X. Construct a sequence $\{Tx_n\}$ such that $Tx_n=S_nx_{n-1}$, $n=1,2,3\cdots$. We can do this since $S_n(X)\subseteq T(X)$. By (1.1),

$$\begin{aligned}d(Tx_{n+1},\ Tx_n,\ a)&=d(S_{n+1}x_n,\ S_nx_{n-1},\ a)\\&\leq q.\max\Big\{d(Tx_n,\ Tx_{n-1},a),\\&\quad d(S_{n+1}x_n,\ Tx_n,\ a),\\&\quad d(S_nx_{n-1},\ Tx_{n-1},a),\\&\quad \frac{1}{2}[d(S_{n+1}x_n,\ Tx_{n-1},\ a)\\&\quad +d(S_nx_{n-1},\ Tx_n,\ a)]\Big\}\\&=q.\max\{d(Tx_n,\ Tx_{n-1},a),\\&\quad d(Tx_{n+1},\ Tx_n,\ a),\\&\quad d(Tx_n,\ Tx_{n-1},\ a),\\&\quad \frac{1}{2}[d(Tx_{n+1},\ Tx_{n-1},\ a)\\&\quad +d(Tx_n,\ Tx_n,\ a)]\},\end{aligned}$$

giving

$$d(Tx_{n+1},\ Tx_n,\ a) \leq q.\max\Big\{d(Tx_n,\ Tx_{n-1},\ a),\ \tfrac{1}{2}d(Tx_{n+1},\ Tx_{n-1}, a)\Big\},$$

and also

$$d(Tx_{n+1},\ Tx_n,\ Tx_{n-1}) = 0.$$

Now, as in (Ram 1982 or Singh-Tiwari-Gupta 1980), it can be shown that $\{Tx_n\}$ is a Cauchy sequence. Since $T(X)$ is complete, it has a limit in $T(X)$. Call it p. Then there exists a point z in X which is a pre-image of p under T, that is $Tz = p$.

Now, for any $n, m,\ n > m$ by (1.1),

$$\begin{aligned} d(S_n x_{n-1},\ S_m z,\ a) &\leq q.\max\Big\{d(Tx_{n-1},\ Tz,\ a), \\ &\quad d(S_n x_{n-1},\ Tx_{n-1},\ a), \\ &\quad d(S_m z,\ Tz,\ a), \\ &\quad \tfrac{1}{2}[d(S_n x_{n-1},\ Tz,\ a) \\ &\quad + d(S_m z,\ Tx_{n-1},\ a)]\Big\} \\ &= q.\max\Big\{d(Tx_{n-1},\ Tz, a),\ 0, \\ &\quad d(S_m z,\ Tz,\ a), \\ &\quad \tfrac{1}{2}[d(Tx_n,\ Tz,\ a) \\ &\quad + d(S_m z,\ Tx_{n-1},\ a)]\Big\}. \end{aligned}$$

Making $n \to \infty$, we obtain

$$d(Tz,\ S_m z,\ a) \leq q \cdot d(S_m z,\ Tz,\ a).$$

Since a is arbitrary,

$$Tz = S_m z.$$

This is true for any m. Hence z is a coincidence point of

T and S_i, $i=1,2,3\cdots$.

Now assume that T and S_i, for each i, commute at z i.e.,

$$TS_iz=S_iTz,\ \ i=1,2,3,\cdots.$$

Also

$$TS_iz=S_iTz=S_iS_jz=S_iS_iz.$$

Then by (1.1), for $m\neq n$,

$$\begin{aligned} d(S_nz,\ S_nS_nz,\ a) &= d(S_nz,\ S_nS_mz,\ a) \\ &= d(S_nz,\ S_mS_nz,\ a) \\ &\leq q.\max\Big\{d(Tz,\ TS_nz,\ a), \\ &\quad d(S_nz,\ Tz,\ a), \\ &\quad d(S_mS_nz,\ TS_nz,\ a), \\ &\quad \frac{1}{2}[d(S_nz,\ TS_nz,\ a) \\ &\quad +d(S_mS_nz,\ Tz,\ a)]\Big\} \\ &= q\cdot d(S_nz,\ S_nS_nz,\ a), \end{aligned}$$

yielding

$$S_nS_nz=S_nz=Tz.$$

So Tz is a fixed point of S_n for every n, naturally. Also, since $TTz=TS_nz=S_nTz=S_nS_nz=S_nz=Tz$, Tz is a fixed point of T. Thus Tz is a fixed point of T and the family $\{S_n\}$. The uniqueness of the common fixed point follows easily.

COROLLARY 3. Let T_1, T_2 and T be mappings from a 2-metric space X to itself such that $T_1(X)\cup T_2(X)\subseteq T(X)$, and for every x, y, a in X

(3.1) $$d(T_1x,\ T_2y,\ a)\leq q.\max\Big\{ d(Tx,\ Ty,\ a),$$

$$d(T_1x,\ Tx,\ a),$$

$$d(T_2y,\ Ty,\ a),$$
$$\frac{1}{2}[d(T_1x,\ Ty,\ a)$$
$$+d(T_2y,\ Tx,\ a)]\Big\}.$$

If $T(X)$ is a complete subspace of X, then T_1, T_2 and T have a coincidence point z. Further if T commutes with each of T_1 and T_2 at z then T, T_1 and T_2 have a common unique fixed point. Indeed Tz is the unique common fixed point.

PROOF. The consequences are immediate if one takes $\{S_n\} = \{T_1, T_2, T_1, T_2, T_1, \cdots\}$ in Theorem 2.

REMARK 4. Corollary 3 improves a number of fixed point theorems for two and three mappings on 1-metric and 2-metric spaces (See, for instance, Singh-Tiwari-Gupta 1980, Singh 1982, Singh-Pant 1983, Singh-Mishra 1983).

As a variant of Theorem 2, we have the following:

THEOREM 5. Let $\{S_n\}$ be a sequence of mappings from a 2-metric space X to itself. Let T be a mapping from X to X satisfying (2.1) and (2.2). If there is a non-negative integer m_i for each S_i such that for all x, y, a of X and for every pair i, j with $i \neq j$,

$$d(S_i^{m_i}x,\ S_j^{m_j}y,\ a) \leq q.\max\Big\{d(Tx,\ Ty,\ a),$$
$$d(S_i^{m_i}x,\ Tx,\ a),$$
$$d(S_j^{m_j}y,\ Ty,\ a),$$
$$\frac{1}{2}[d(S_i^{m_i}x,\ Ty,\ a)$$
$$+d(S_j^{m_j}y,\ Tx,\ a)]\Big\}$$

Then T and $S_i^{m_i}(i=1, 2, 3, \cdots)$ have a coincidence point z. Further if T and $S_i^{m_i}(i=1, 2, 3, \cdots)$ commute at z then T and the sequence $\{S_n\}$ of mappings have a unique common fixed point.

PROOF. Clearly $S_i^{m_i}(X) \subseteq S_i(X) \subseteq T(X)$. Thus Theorem 2 pertains to $S_i^{m_i}$ and T. So there is a unique point p in $T(X)$ such that

$$p = Tz = S_i^{m_i} z$$

and

$$S_i^{m_i} p = Tp = p, \quad i = 1, 2, 3, \cdots.$$

From this

$$S_i p = S_i S_i^{m_i} p = S_i^{m_i} S_i p.$$

This shows that $S_i p$ is a fixed point of T and $S_i^{m_i}$. The uniqueness of p implies that

$$p = Sp = S_i p.$$

REMARK 6. Results of Lal-Singh(1978), Rhoades(1979), Ram(1982) and Singh-Tiwari-Gupta(1980) may be obtained as corollaries. In particular, an improved version of Corollary 1 of Lal-Singh(1978) is obtained when T is an identity mapping in the above theorem.

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COHERENT POLYNOMIAL RINGS OVER REGULAR P. I. RINGS

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1. Introduction

J. P. Soublin [11] showed that the analog of the Hilbert basis theorem fails for coherent rings, but the ring of polynomials in an indeterminate over a commutative von Neumann regular ring with identity is coherent. G. Sabbagh [10] has proved that the ring of polynomials in any finite number of indeterminates over a commutative von Neumann regular ring with identity is coherent.

In this paper, the following result will be studied if R is a von Neumann regular P. I. ring then $R[x_1, x_2, \cdots, x_n]$ is a coherent ring. Also, we obtain the following corollaries; (1) If R is a commutative von Neumann regular ring, then $R[x_1, x_2, \cdots, x_n]$ is a coherent ring, see G. Sabbagh [10]. (2) Let R be a P. I. ring. Then R is von Neumann regular if and only if $R[x]$ is semihereditary.

2. Self-injective ring and Azumaya algebra

We introduce P. I. rings and Azumaya algebras that will be used in the study of our main results. It is known that, since singular ideals of semiprime P. I. ring is zero [3], maximal quotient rings of semiprime P. I. rings are regular

with identity [2], self-injective [2] and P.I. [7]. By combining the above facts and Armendariz's two decomposition theorems [1], we obtain the fact that the maximal quotient ring of a regular P.I. ring is Azumaya.

THEOREM 2.1[1]. If R is a regular self-injective ring with a P.I., then $R \cong \prod_{\lambda \in \Lambda} R_\lambda$ where each R_λ is a matrix ring over a strongly self-injective ring with a P.I..

THEOREM 2.2[1]. Let R be a regular self-injective ring with a P.I., then $R = \oplus \sum_{i=1}^{k} R_i$ where R_i is either zero or else each R_i is a product of regular self-injective rings each of which is stable of degree i, $1 \leqq i \leqq k$.

The conclusion of Theorem 2.2 obtains following.

COROLLARY 2.3. If R is a regular P.I. ring, then the quotient ring $Q(R)$ of R is an Azumaya algebra.

PROOF. If R is a regular P.I. ring, then $Q(R)$ is a regular self-injective P.I. ring. By Theorem 2.2,

$$Q(R) = \oplus \sum_{i=1}^{k} Q_i(R)$$

where $Q_i(R)$ is a product of regular self-injective rings each of which is stable of degree $1 \leqq i \leqq k$. Hence each $Q_i(R)$ is stable unital semiprime P.I. ring. By Procesi, stable unital semiprime P.I. rings are Azumaya algebras. Therefore each $Q_i(R)$ is an Azumaya algebra. Thus $Q(R) = \oplus \sum_{i=1}^{k} Q_i(R)$ is an Azumaya algebra.

3. Coherent Ring

By Sabbagh, the ring of polynomials in any number of

indeterminates over a commutative von Neumann regular ring with identity is coherent. In this section, we obtain similar results for a von Neumann regular P.I. ring.

THEOREM 3.1 [6]. R is a subdirect of the rings S_i, $i \in I$, if and only if $S_i \cong R/K_i$, K_i an ideal of R, and $\bigcap_{i \in I} K_i = 0$.

THEOREM 3.2 [6]. Every left R-module is flat if and only if R is regular.

THEOREM 3.3. Let R be a commutative von Neumann regular ring and $S = \Pi R/P_i$, P_i a prime ideal of R. Then $S[x]$ is faithfully flat over $R[x]$.

PROOF. First note that as left S-module

$$S[x] \cong S \otimes_R R[x].$$

Next note that, for any left $S[x]$-module M, there are the following left S-module isomorphisms

$$S[x] \otimes_{R[x]} M \cong S \otimes_R R[x] \otimes_{R[x]} M \cong S \otimes_R M.$$

Now consider an exact sequence of $R[x]$-module

$$0 \to M \to N.$$

Then, since S_R is flat, the following diagram is commutative and its columns are isomorphisms

$$\begin{tikzcd} 0 \arrow[r] & S \otimes_R M \arrow[r] \arrow[d, phantom, "\wr\|"] & S \otimes_R N \arrow[d, phantom, "\wr\|"] \\ 0 \arrow[r] & S[x] \otimes_{R[x]} M \arrow[r] & S[x] \otimes_{R[x]} N \end{tikzcd}.$$

Thus $S[x]$ is flat. Now to show that $S[x]$ is faithfully flat as $R[x]$-module, let $S[x] \otimes_{R[x]} M = 0$. Then $S \otimes_R M = 0$ so that $M = 0$. Hence $S[x]$ is faithfully flat over $R[x]$.

COROLLARY 3.4. Let R be a commutative von Neumann

regular ring. Then $S[x_1, x_2, \cdots, x_n] = \Pi R/P_i[x_1, x_2, \cdots, x_n]$ is faithfully flat over $R[x_1, x_2, \cdots, x_n]$.

THEOREM 3.5. Let R be a regular P.I. ring. Then the maximal quotient ring $Q(R)$ of R is faithfully flat over R.

PROOF. Since $Q(R)$ is regular, it is flat. Let M be a left R-module. Suppose that $Q(R)\otimes_R M=0$, using the flatness of M as a left R-module $M \cong R\otimes_R M \leqq Q(R)\otimes_R M = 0$. Thus $M=0$.

COROLLARY 3.6. If R is regular P.I. ring, then $Q(R)[x_1, x_2, \cdots, x_n]$ is faithfully flat over $R[x_1, x_2, \cdots, x_n]$.

THEOREM 3.7 [10]. Let R be a subring of the right coherent ring S such that S is faithfully left flat over R. Then R is a right coherent ring.

THEOREM 3.8. If R is a regular P.I. ring, then $R[x_1, x_2, \cdots, x_n]$ is a coherent ring.

PROOF. Since the maximal quotient ring $Q(R)$ of R is a regular self-injective P.I. ring, $Q(R) = \sum_{i=1}^{n} Q_i(R)$, where each $Q_i(R)$ is an Azumaya algebra. Therefore,

$$Q(R)[x_1, x_2, \cdots, x_n] = \sum_{i=1}^{n} Q_i(R)[x_1, x_2, \cdots, x_n]$$

since $Q_i(R) \cong Mat_n(A)$, where A is regular in which every idempotent is a central idempotent. For each i,

$$Q_i(R)[x_1, x_2, \cdots, x_n] = Mat_n(A[x_1, x_2, \cdots, x_n]).$$

Now we show that $A[x_1, x_2, \cdots, x_n]$ is coherent, since matrix ring with coefficient in a coherent ring is also coherent [9]. Let $\{P_i | i \in I\}$ be a set of all prime ideals of A. Then $\bigcap_i P_i = 0$. Therefore R is a subdirect product of division

rings $\{A/P_i \mid i \in I\}$. So $\Pi A/P_i$ is faithfully flat over A. It follows that $\Pi A/P_i[x_1, x_2, \cdots, x_n]$ is faithfully flat over $A[x_1, x_2, \cdots, x_n]$. Since A/P_i is a division ring, $A/P_i[x_1, x_2, \cdots, x_n]$ is coherent. Hence $\Pi A/P_i[x_1, x_2, \cdots x_n]$ is coherent. Thus

$$\begin{array}{ccc} \Pi A/P_i[x_1,x_2,\cdots,x_n] & \cong (\Pi A/P_i)[x_1,x_2,\cdots,x_n] & : \text{right coherent} \\ \cup\!\!\mid & \text{faithfully flat} \quad \cup\!\!\mid & \\ A[x_1,x_2,\cdots,x_n] & \longrightarrow A[x_1,x_2,\cdots,x_n] & : \text{coherent.} \end{array}$$

Therefore $R[x_1, x_2, \cdots, x_n]$ is coherent.

The corollary of ours is proved by Sabbagh [21].

COROLLARY 3.9. If R is commutative regular, then $R[x_1, x_2, \cdots, x_n]$ is coherent.

The weak dimension of an R-module will be denoted by $WD(R)$. Regular rings are characterized by nullity of the weak dimension. From properties of semihereditary and coherent ring, we see that a ring is left semihereditary if and only if it is left coherent and has weak dimension at most one.

COROLLARY 3.10. Let R be a P.I. ring. Then R is von Neumann regular if and only if $R[x]$ is semihereditary.

PROOF. In the proof of Theorem 3.8, we see that for the maximal quotient ring $Q(R)$ of a ring R, $Q(R)[x]$ is coherent. By Theorem 3.7 and Lemma 3.8, $R[x]$ is coherent. For every ring R, $WD(R) \leqq WD(R[x]) \leqq 1 + WD(R)$ since $WD(R) = 0$, $0 \leqq WD(R[x]) \leqq 1$. If $WD(R[x]) = 0$, then $R[x]$ is semi-simple Artinian. Hence $WD(R[x]) \leqq 1$.

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GALOIS SUBRINGS OF UTUMI RINGS OF QUOTIENTS

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1. Introduction

For the Galois theory of semiprime rings the importance of crossed products was firstly noticed by T. Nakayama in [8]. After his proving the beautiful normal basis theorem crossed product has greatly influenced to Galois theory. But still at his time there was no systematic scheme between crossed products and Galois subrings. M. Cohen [3] successfully provided a systematic "Morita context" between them by the hint of Chase, Harrison and Rosenberg. But actually J. Osterburg and J. Park pointed out that her context is the derived context of some of module.

As J. Osterburg and J. Park did in [9] we consider crossed products and Galois subrings altogether at the same time via the drived Morita context. Indeed we prove that the Utumi quotient ring of Galois subring is the Galois subring of the Utumi quotient ring in different way. Also we consider the normal basis theorem for regular self-injective ring case. By our normal basis theorem we can generalize M. Cohen's result [3] for the semi-simple artinian ring case. Some our results in this paper were alrealy proved in [9].

2. Preliminaries

In this section some necessary definitions and properties will be given. All rings are assumed to have identity element. R denotes a ring and all modules are right R-modules, unless otherwise mentioned.

DEFINITION 2.1. A *Morita context* consists of two rings R and S, two bimodules ${}_SP_R$ and ${}_RQ_S$, and two bimodule homomorphisms (called the pairings)

$$(\, , \,): Q\otimes_S P \longrightarrow R$$

and

$$[\, , \,]: P\otimes_R Q \longrightarrow S$$

satisfying the associativity conditions $q[p, q'] = (q, p)q'$ and $p(q, p') = [p, q]p'$.

The images of the pairings are called the *trace ideals* of the context, and are denoted by T_R and T_S. We abbreviate a context by the symbol $\langle P, Q\rangle$.

For any R-module P_R let $P^* = Hom(P_R, R_R)$ and $S = End(P_R)$. Then P^* is a right S and left R-bimodule. Define pairings $(\, , \,)$ and $[\, , \,]$ as follows; $(\, , \,): P^*\otimes_S P \longrightarrow R$ by $(f, p) = f(p)$ and $[\, , \,]: P\otimes_R P^* \longrightarrow S$ by $[p, f](x) = pf(x)$ for x in P. Then it can be easily checked that (R, P, P^*, S) is a Morita context between R and S. This particular context is called the *derived Morita context* of P_R. For the left module case, we can define the derived Morita context similarly.

EXAMPLE 2.2. From two module X_A and T_A, define

$$R = End_A\,(X),\ S = End_A\,(Y),$$
$$P = Hom_A(X, Y), \text{ and } Q = Hom_A(Y, X)$$

with pairings by composition. Then $\langle P, Q\rangle$ is the Morita context of two rings R and S.

EXAMPLE 2.3. Let $\langle A, B, P, Q, \alpha, \beta\rangle$ be a Morita context. Define the generalized matrix ring

$$R=\begin{bmatrix} A & P \\ Q & B \end{bmatrix},$$

α, β by using ordinary matrix addition and multiplication by means of α and β. Then R is actually a ring.

Put
$$e=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then
$$A=eRe, \qquad B=(1-e)R(1-e),$$
$$P=eR(1-e), \qquad Q=(1-e)Re.$$

In general R is an arbitary ring with an idempotent e, then $\langle eRe,\ (1-e)R(1-e),\ eR(1-e),\ (1-e)Re,\ \alpha,\ \beta\rangle$ is a Morita context for suitable α, β.

Associated with any Morita context $\langle P, Q\rangle$ there are eight natural maps, e.g., $p\in P\to[p, -]\in Q^*=Hom_S(Q, End_R(P))$ and $r\in R\to(q\to rq)\varepsilon End_S(Q)$, where $Q=Hom_R(P, R)$ and $S=End_R(P)$.

DEFINITION 2.4. The context $\langle P, Q\rangle$ is called *non-degenerate* if all these natural maps are injective.

DEFINITION 2.5. A Morita context is *right normalized* if the four natural maps $P\to Q^*$, $Q\to P^*$, $R=End_S(Q)$ and $S\to End_R(P)$ are isomorphisms.

THEOREM 2.6 [7, Theorem 19]. If a Morita context $\langle P, Q\rangle$ between two rings R and S is nondegenerated, then the maximal quotient Morita context $\langle P, Q\rangle$ between $Q_{\max}(R)$

and $Q_{\max}(S)$ induced by $\langle P, Q\rangle$ is right normalized.

Let A be a finite dimensional central simple algebra over its center field F. Then A has a maximal subfield K. If K is a Galois extension of F with a Galois group G then there are invertible elements $\{a_g: g \text{ in } G\}$ of A such that $A=\bigoplus Ka_g$, a direct sum of left K-vector spaces, where for all x in K, $xa_g=a_g g(x)$. Moreover defining $t(g, h)=a_g a_h a_{gh}^{-1}$ for each g, h in G, we have $t(g, h)$ is in K and the following equations hold for all g, h, k in G:

$$t(g, h)t(gh, k)=t(h, k)t(g, k).$$

By this fact we define a crossed product formally as follows.

DEFINITION 2.7. Let R be a ring and G be a group. Given a group homomorphism $p:G\to\mathrm{Aut}(R)$ and a map $t:G\times G\to U(R)$ the units of R such that

(1) $\quad t(x, y)t(xy, z)=t(y, z)^{p(x)}t(x, yz)$

and

(2) $\quad t(x, y)a^{p(xy)}=a^{p(x)p(y)}t(x, y)$

for all x, y, z in G and a in R. We define the *crossed product* $R*G$ to be the set of all formal sums of the form $\sum a_x\bar{x}$ with a_x in R and $a_x=0$ for almost all x in G. The addition in $R*G$ is defined componentwise and the multiplication is given by the rule

$$(a_x\bar{x})(a_y\bar{y})=a_x a_y^{p(x)}t(x, y)\overline{xy}.$$

This makes $R*G$ an associative ring with identity $t(1, 1)^{-1}\bar{1}$. When $t(x, y)=1$ for every x, y in G, the crossed product is called a *skew group ring* and denoted by RG.

EXAMPLE 2.8. Let G be a finite group of automorphisms of a field F. Then the skew group ring FG is the $n \times n$ matrix ring over the fixed field F^G, where n is the order of G.

We set some notations and basic definitions. Let G be a group of automorphisms acting on R. By r^g we mean the image of r under g in G. The fixed ring of R is $R^G = \{r$ in $R \mid r^g = r$ for all g in $G\}$. The *trace* of x is $tr(x) = \sum x^g$. Note that $tr(x)$ is in R^G. An ideal L(left or two sided) of R is called G-invariant if L^g is contained in L for all g in G. The ring R is said to have no $|G|$-torsion if $|G|r = 0$ for r in R implies that $r = 0$. If $\mathscr{F}$ is a Gabriel filter on R, then $t(R_R)$ is the set of all a in R whose right annihilator is a member of $\mathscr{F}$ and is called the torsion submodule of R_R with respect to $\mathscr{F}$.

3. Main Results

When G is a finite group of automorphisms of R we can form a skew group ring $S = RG$ over R.

For a given Gabriel filter $\mathscr{F}$ on R, $\overline{\mathscr{F}} = \{\bar{D} \triangleleft_r RG \mid \bar{D} \cap R$ is in $\mathscr{F}\}$ is a Gabriel filter on RG by K. Louden [6, Lemma 8].

We recall that $t(R_R)$ is the torsion submodule of R_R with respect to $\mathscr{F}$, $t(S_R)$ is the torsion submodule of S_R with respect to $\mathscr{F}$ and $t(S_S)$ is the torsion submodule of S_S with respect to $\overline{\mathscr{F}}$. Then we can obtain $t(R_R)S = t(S_S)$ and $t(S_R) = t(S_S)$. Furthermore $t(R_R) = t(S_R) \cap R$. We call that $\mathscr{F}$ is a G-invariant (or an automorphism invariant) if I^g is an

element of $\mathscr{F}$ for all I in $\mathscr{F}$, g in G.

LEMMA 3.1. If $\mathscr{F}$ is an automorphism invariant, then every automorphism of R can be extended to an automorphism of $Q(R)$.

PROOF. Let $t(R)$ be the torsion submodule of R with respect to $\mathscr{F}$. Since $\mathscr{F}$ is an automorphism invariant, $g(t(R))=t(R)$ for all g in Aut (R). Let f be an R-homomorphism from D to $R/t(R)$ which represents an element of $Q(R)$. Define $g(f)$ from $g(D)$ to $R/t(R)$ by $g(f)\,(g(d))=g(f(d))$ for d in D. Then $g(f(R))=t(R)g(f)$ is well-defined. For, if $f(d_1)=f(d_2)$ for $f(d_1)$, $f(d_2)$ in $R/t(R)$. Put $f(d_1)=r_1+t(R)$ and $f(d_2)=r_2+t(R)$ for r_1, r_2 in R. Since $r_1+t(R)=r_2+t(R)$ we have r_1-r_2 is in $t(R)$ and hence $g(r_1-r_2)$ is in $t(R)$. It follows that $g(f(d_1))=g(f(d_2))$. Therefore $g(f)(g(d_1))=g(f)(g(d_2))$ which completes the well-definedness. Next we show that $g(f)$ is a homomorphism. Since $g(f)(g(d_1)+g(d_2))=g(f)(g(d_1+d_2))=g(f(d_1+d_2))=g(f(d_1)+f(d_2))=g(f(d_1))+g(f(d_2))=g(f)g(d_1)+g(f)g(d_2)$ and $g(f)(g(d_1)r)=g(f)(g(d_1r))=g(f(d_1r))=g(f(d_1)r)=g(f(d_1))r=g(f)(g(d_1))r$ for all $g(d_1)$, $g(d_2)$ in $g(D)$ and r in R. Thus $g(f)$ is a homomorphism and this defines g on $Q(R)$ to be an automorphism.

By Lemma 3.1, if $\mathscr{F}$ is a G-invariant filter, then G can be considered as an automorphism group on $Q_{\mathscr{F}}(R)$. Let $[q]$ be in $Q_{\mathscr{F}}(R)$ which is represented by $q: D\to R/t(R)$ with D in $\mathscr{F}$. Define $\bar{q}$: $DS\to S/t(S)$ by $\bar{q}(\sum d_g g)=\sum q(d_g)g$. Then since $R/t(R)$ is contained in $S/t(S_R)$ and $t(S_R)=t(S_S)$, $\bar{q}$ is well-defined and an S-homomorphism and $\bar{q}|D=q$ with

DS in $\overline{\mathscr{F}}$. Let $[\bar{q}]$ be in $Q_{\overline{\mathscr{F}}}(S)$ represented by this map $\bar{q}$. For any g in G, the left multiplication $L_g: S_S \longrightarrow S_S$ induces an S-homomorphism $\bar{L}_g: S/t(S_S) \longrightarrow S/t(S_S)$. So $\bar{L}_g\bar{q}$: $DS \longrightarrow S/t(S_S)$ represents an element $[\bar{L}_g\bar{q}]$ in $Q_{\overline{\mathscr{F}}}(S)$.

THEOREM 3.2. Let G be a finite group of automorphisms of R and $S=RG$. Then for a G-invariant filter $\mathscr{F}$ on R, $Q_{\mathscr{F}}(R)G$ is isomorphic to $Q_{\overline{\mathscr{F}}}(S)$.

PROOF. Define f: $Q_{\mathscr{F}}(R)G \longrightarrow Q_{\overline{\mathscr{F}}}(S)$ by $f(\sum g[q_g]) = \sum[\bar{L}_g\bar{q}_g]$. We divide the proof into five steps.

STEP 1. We show that f is well-defined.

If $\sum g[q_g] = \sum g[w_g]$, then $[q_g]=[w_g]$ for all g in G. Thus there exists D_g in $\mathscr{F}$ such that q_g and w_g are coincided on D_g for all g in G. Therefore for all g in G, $\bar{q}_g$ and $\bar{w}_g$ agree on D_gS and D_gS is in $\overline{\mathscr{F}}$. So for all g in G, $\bar{L}_g\bar{q}_g = \bar{L}_g\bar{w}_g$ on D_gS. Hence $[\bar{L}_g\bar{q}_g]=[\bar{L}_g\bar{w}_g]$ for all g in G. Consequently $\sum[\bar{L}_g\bar{q}_g] = \sum[\bar{L}_g\bar{w}_g]$. Hence $f(\sum g[q_g]) = f(\sum g[w_g])$ and therefore f is well-defined.

STEP 2. We show that f is additive.

Since $[L_g][\bar{q}_g] = [L_g\bar{q}_g]$ and $\overline{[q_g+w_g]} = [\bar{q}_g]+[\bar{w}_g]$, we have following:

$$\begin{aligned}
& f(\sum g[q_g]+g[w_g]) = f(\sum g([q_g]+[w_g])) \\
&= f(\sum g[q_g+w_g]) = \sum[\bar{L}_g\overline{(q_g+w_g)}] = \sum[\bar{L}_g]\overline{[q_g+w_g]} \\
&= \sum[\bar{L}_g]([\bar{q}_g]+[\bar{w}_g]) = \sum[\bar{L}_g][\bar{q}_g] + \sum[\bar{L}_g][\bar{w}_g] \\
&= \sum[\bar{L}_g\bar{q}_g] + \sum[\bar{L}_g\bar{w}_g] = f(\sum g[q_g]) + f(\sum g[w_g]).
\end{aligned}$$

Hence f is additive.

STEP 3. We show that f is a homomorphism.

Choose $g[q_g]$ and $h[w_h]$ in $Q_{\mathcal{F}}(R)G$ with $q_g: D_1 \longrightarrow R/t(R)$ and w_h: $D_2 \longrightarrow R/t(R)$ for g, h in G. Then we have $g[q_g]$ $h[w_h]=gh[q_g]^h[w_h]$. Let $[x]=[q_g]^h$. Then $[x]$ is represented by $x: D_1^h \longrightarrow R/t(R)$; $x(d_1^h)=q_g(d_1)^h$ for d_1 in D_1. So $g[q_g]h[w_h]=gh[x][w_h]$. Let $y=[xw_h]$. Then $[y]$ is represented by y; $w_h^{-1}(D_1^h/t(D_1^h)) \xrightarrow{w_h} D_1^h/t(D_1^h) \xrightarrow{\bar{x}} R/t(R)$, where $\bar{x}$ is the induced R-homomorphism from x. Hence $f(g[q_g]h[w_h])=f(gh[y])=[\bar{L}_{gh}\bar{y}]$ is represented by S-homomorphism: $w_h^{-1}(D_1^h/t(D_1^h))S \xrightarrow{\bar{y}} S/t(S) \xrightarrow{\bar{L}_{gh}} S/t(S)$.

On the other hand, $f(g[q_g])f(h[w_h])=[\bar{L}_g\bar{q}_g][\bar{L}_h\bar{w}_h]$ is represented by the composition; $\bar{w}_h^{-1}\bar{h}^{-1}(D_1S/t(D_1S)) \xrightarrow{\bar{w}_h}$ $\bar{L}_h^{-1}(D_1S/t(D_1S)) \xrightarrow{\bar{L}_h} D_1S/t(D_1S) \xrightarrow{\bar{\bar{q}}_g} S/t(S) \xrightarrow{\bar{L}_g} S/t(S)$ where $\bar{\bar{q}}_g$ is the induced map from $\bar{q}_g$. In this case $\bar{h}^{-1}(D_1S/t(D_1S))$ $=D_1^hS/t(D_1^hS)$ and $\bar{w}_h^{-1}\bar{L}_h^{-1}(D_1S/t(D_1S))=w_h^{-1}(D_1^hS/t(D_1^hS))=w_h^{-1}(D_1^h/t(D_1^h))S$. Let $D_3=w_h^{-1}(D_1^h/t(D_1^h))$. Then for d in D_3 and k in G, $(\bar{L}_{gh}\bar{y})(dk)=\bar{L}_{gh}(y(d)k)=$ $\bar{L}_{gh}y(d)k=\bar{L}_{gh}[\bar{x}w_h(d)]k$. Let $w_h(d)=d_1^h+t(D_1^h)$ with d_1 in D_1. Then $\bar{L}_{gh}[\bar{x}w_h(d)]k=\bar{L}_{gh}[\bar{x}(d_1^h)+t(D_1^h)]k=\bar{L}_{gh}[x(d_1^h)]$ $k=\bar{L}_{gh}[q_g(d_1)^h]k=\bar{L}_g[q_g(d_1)]hk$. And we have $(\bar{L}_g\bar{\bar{q}}_g\bar{L}_hw_h)$ $(dk)=(\bar{L}_g\bar{\bar{q}}_g\bar{L}_h)(w_h(d)k)=(\bar{L}_g\bar{\bar{q}}_g\bar{L}_h)(d_1^h+t(D_1^h))k=(\bar{L}_g\bar{\bar{q}}_g)$ $(hd_1^hk+t(S))=(\bar{L}_g\bar{\bar{q}}_g)(d_1hk+t(S))=\bar{L}_g(q_g(d_1)hk)=\bar{L}_g[q_g$ $(d_1)]hk$. Hence $f(g[q_g]h[w_h])=f(g[q_g])f(h[w_h])$. Therefore f is a homomorphism.

STEP 4. We show that f is one to one.

Suppose $[\sum \bar{L}_g\bar{q}_g]=0$ with $\bar{q}_g: D_gS \longrightarrow S$ with D_g in $\mathcal{F}$, g in G. Then $\sum[\bar{L}_g\bar{q}_g]$ is represented by the S-homomorphism: $\cap(D_gS) \longrightarrow S/t(S)$; $x \longmapsto \sum \bar{L}_g(\bar{q}_g(x))$. Since $\sum[\bar{L}_g\bar{q}_g]$

$=0$, there exists a G-invariant D_0 in $\mathscr{F}$ such that D_0S is contained in $\bigcap (D_gS)$ and $\bar{L}_g(q_g(x))=0$ on D_0S. Now for every d_0 in D_0, $0=\bar{L}_g(\bar{q}_g(d_0)) = \sum \bar{L}_g(q_g(d_0))$. Let $q_g(d_0)= r_g+t(R_R)$ with r_g in R. Then $0=\sum \bar{L}_g(q_g(d_0))=\sum \bar{L}_g(r_g+ t(R))=\sum \bar{L}_g(r_g+t(S)) = \sum (gr_g+t(S))=(\sum r_g^{g-1}g)+t(S)$. Hence $\sum r_g^{g-1}g$ is a element of $t(S)=t(R)S$. So for all g in G, r_g^{g-1} is contained in $t(R)$. Hence r_g is a element of $t(R)$ for all g in G. Therefore $q_g(d_0)=r_g+t(R)=0$ for all g in G and d_0 in D_0. Hence $q_g=0$ on D_0.

Therefore $[q_g]=0$ and so $\sum g[q_g]=0$. Hence Ker $f=\{0\}$. Thus f is $1-1$.

STEP 5. We show that f is onto.

Let $[x]\in Q_{\mathscr{F}}(S)$ represented by x: $DS\longrightarrow S/t(S)$ with D in $\mathscr{F}$. Let p be a map from $S/t(S)$ to $R/t(R)$ defined by $p(\sum r_h h+t(S))=r_g^g+t(R)$ for all g in G. Since $t(R_R)S= t(S_S)$, p is well-defined and an R-homomorphism.

$$\begin{tikzcd} D_R \arrow[r, "x"] \arrow[dr, "px"'] & S/t(S) \arrow[d, "p"] \\ & R/t(R) \end{tikzcd} \qquad \begin{tikzcd} d \arrow[r, mapsto] \arrow[dr] & r(d)_h^h + t(S) \arrow[d] \\ & r(d)_g^g + t(R_R) \end{tikzcd}$$

where $r(d)_h$ is a element of R for all h in G. Thus px is an R-homomorphism: $D_R\longrightarrow R/t(R)$. We will show that $[x]=\sum[\bar{L}_g\, \bar{p}\bar{x}]=f(\sum g[px])$. $\sum[\bar{L}_g\, \bar{p}\bar{x}]$ is represented by $DS \longrightarrow S/t(S)$; $y\longmapsto \sum \bar{L}_g[(px)(y)]$. Now for d in D and h in G; $\sum \bar{L}_g[\bar{p}\bar{x}(dh)]=\sum \bar{L}_g([(px)(d)]h)=\sum(\bar{L}_g[(px)(d)])h= \sum \bar{L}_g[r(d)_g^g+t(R)]h=\sum \bar{L}_g(r(d)_g^g+t(S))h = \sum (gr(d)_g+t(S))h=\sum(r(d)_g^g g+t(S))h=\sum(r(d)_g^g+t(S))gh$ and $x(dh)=$

$x(d)h=\sum(r(d)_g g+t(S))h = \sum(r(d)_g+t(S))gh$. Hence $[x]=\sum[\bar{L}_g\bar{p}\bar{x}]=f(\sum g[\bar{p}\bar{x}])$. Thus f is onto. Resultly f is an isomorphism. Thus $Q_{\mathcal{F}}(R)G$ is isomorphic to $Q_{\overline{\mathcal{F}}}(S)$.

An overring S of a ring R with same identity is called a *finite normalizing extension ring* of R if S is finitely generated as an R-module by elements which normalize R, that is, $S=\sum_{i=1}^{n} R x_i$ with $Rx_i=x_iR$ for each i.

For example, it includes crossed product $R*G$ with a finite group G.

LEMMA 3.3 [5, Theorem 3.2]. If $S=\sum_{i=1}^{n} x_iR$ is a finite normalizing extension of R with $X_1=1_R=1_S$, then for M in *Mod-R*, $Hom_R(S_R,M_R)$ is an injective S-module if and only if M is an injective R-module.

Immediately by the above Lemma we can get $Hom_R(S_R, E_R(R))=E_S(Hom_R(S_R, R_R))$.

LEMMA 3.4. Let $S=RG$ be a skew group ring with a finite group G, and let $\mathcal{F}$ be the Lambek topology (or topology) on R. Then $\overline{\mathcal{F}}=[I\lhd_r RG: I\cap R\in\mathcal{F}\}$ is the Lambek topology on RG.

PROOF. By the above Lemma 3.3, we have $Hom_R(RG, E(R)_R)=Hom_{RG}(Hom_R(RG, R_R))=E(RG)_{RG}$. Since $E(R)_R$ is an injective cogenerator, $Hom_R(RG, E(R))=E(RG)_{RG}$ is an injective cogenerator by K. Louden [6, Proposition 4]. Therefore $\overline{\mathcal{F}}$ is also a Gabriel filter on RG.

COROLLARY 3.5. $Q_{\max}(R)G$ is isomorphic to $Q_{\max}(RG)$.

PROOF. Define $f\colon Q_{\max}(R)G \longrightarrow Q_{\max}(RG)$ by $f(\sum g[q_g]) = \sum[L_g\bar{q}_g]$.

STEP 1. f is well-defined.

It is obvious that $\sum[L_g\bar{q}_g]$ is in $Q_{\max}(S)$. Now if $\sum g[q_g] = \sum g[w_g]$ with $[q_g]$, $[w_g]$ in $Q_{\max}(R)$. Then we can obtain $[q_g]=[w_g]$ for all g in G. Therefore there exists D_g in $\mathscr{F}$ such that $q_g|D_g = w_g|D_g$ for all g in G. Thus we have $\bar{q}_g|D_gS = \bar{w}_g|D_gS$ and D_gS in $\bar{\mathscr{F}}$ by Lemma 3.4 for all g in G. Hence $L_g\bar{q}_g|D_gS = L_g\bar{w}_g|D_gS$ for all g in G. Therefore we have $[L_g\bar{q}_g] = [L_g\bar{w}_g]$ for all g in G. Consequently, $f(\sum g[q_g]) = \sum[L_g\bar{q}_g] = \sum[L_g\bar{w}_g] = f(\sum g[w_g])$. Thus f is well-defined.

STEP 2. f is additive.

Since $[L_g]\,[\bar{q}_g] = [L_g\bar{q}_g]$ and $\overline{[q_g+w_g]} = [\bar{q}_g]+[\bar{w}_g]$, f is clearly additive.

STEP 3. f is a homomorphism.

Suppose $f(g[q_g]) = [L_g\bar{q}_g]$ with $q_g\colon D_1 \longrightarrow R_R$ and $f(h[w_h]) = [L_h\bar{w}_h]$ with $w_h\colon D_2 \to R_R$. Then $f(g[q_g]\,h[w_h]) = f(gh[q_g]^h[w_h])$. Let $[u]=[q_g]^h$. Then $[u]$ is represented by $u\colon D_1{}^h \to R_R$; $u(d_1{}^h) = q_g(d_1)^h$ for d_1 in D_1. So $f(g[q_g]h[w_h]) = f(gh[u][w_h])$. Let $[v]=[u][w_h]$ and let $D_3 = w_h{}^{-1}(D_1{}^h)$. Then $v\colon D_3 \xrightarrow{w_h} D_1 \xrightarrow{u} R_R$ represents $[v]$. So $f(g[q_g]h[w_h]) = f(L_{gh}[v]) = [L_{gh}\bar{v}]$; $D_3 \xrightarrow{\bar{v}} S_S \xrightarrow{L_{gh}} S_S$. Now for $f(g[q_g])f(h[w_h]) = [L_g\bar{q}_g]\,[L_h\bar{w}_h]$ is represented by the composition; $w_h{}^{-1}L_h{}^{-1}(D_1S) \xrightarrow{\bar{w}_h} L_h{}^{-1}(D_1S) \xrightarrow{L_h} D_1S \xrightarrow{\bar{q}_g} S_S \xrightarrow{g} S_S$. In this case $L_h{}^{-1}(D_1S) = D_1{}^hS$ and $\bar{L}_h{}^{-1}(L_h{}^{-1}(D_1S)) = \bar{L}_h{}^{-1}(d_1{}^hS) = L_h{}^{-1}(D_1{}^h)S = D_3S$. Now for dk in D_3S with d in D_3 and k in G;

$(L_{gh}\bar{v})(dk)=gh(v(d)k)=gh\ u(w_h(d))k=gh\ u[(w_h(d)^{h^{-1}})^h]k$
$=gh[q_g(w_h(d)^{h^{-1}})]^h k=g[q_g(w_h(d)^{h^{-1}})]hk.$

On the other hand $(L_g\bar{q}_g L_h\bar{w}_h)(dk)=(L_g\bar{q}_g L_h)(w_h(d)k)=$ $(L_g\bar{q}_g)(hw_h(d)k)=(L_g\bar{q}_g)(w_h(d)^{h^{-1}}hk)=L_g(\bar{q}_g(w_h(d)^{h^{-1}}hk))$ $=g(q_g(w_h(d)^{h^{-1}})hk)=g(q_g(w_h(d)^{h^{-1}}))hk$. Hence $L_{gh}\bar{v}|D_3S=$ $L_g\bar{q}_g L_h\bar{w}_h|D_3S$. Therefore

$$f(g[q_g]h[w_h])=f(g[q_g])\,f(h[w_h]).$$

So f is a homomorphism.

STEP 4. f is one to one.

Suppose $\sum[L_g\bar{q}_g]=0$ with $\bar{q}_g:D_gS_S\longrightarrow S_S$. Then $\sum[L_g\bar{q}_g]$ is represented by S-homomorphism;

$$\bigcap(D_gS)\longrightarrow S_S.$$
$$x\longrightarrow\sum g(\bar{q}_g(x)).$$

Since $\sum[L_g\bar{q}_g]=0$, there exists G-invariant D_0 in $\mathscr{F}$ and $D_0S\subseteq\bigcap(D_gS)$ such that $\sum g(\bar{q}_g(x))=0$ for all x in D_0S. Now for d_0 in D_0, $0=\sum g(\bar{q}_g(d_0))=\sum q_g(d_0)^{g^{-1}}g$. Hence $q_g(d_0)^{g^{-1}}=0$ and so $q_g(d_0)=0$ for all g in G and d_0 in D_0. So $q_g=0$ on D_0, $q_g=0$ on D_g for all g in G. So $[q_g]=0$. Hence we have $\sum g[q_g]=0$ for all g in G. Therefore $\mathrm{Ker} f=\{0\}$. Thus f is one to one.

$$\begin{tikzcd} D_R \arrow[r, "x"] \arrow[dr, "px"'] & S_R \arrow[d, "p"] \\ & R_R \end{tikzcd}$$

STEP 5. f is onto.

Let $[x]$ be element of $Q_{max}(S)$ represented by $x: DS_S\rightarrow S_S$ with D in $\mathscr{F}$.

Define p: $S_R\rightarrow R_R$ by $p(\sum r_h h)$ $=r_g{}^g$ for all g in G.

Then p is an R-homomorphism. So $x(d)=\sum[(px)(d)]^{g^{-1}}g$ for all d in D. So $[px]$ in $Q_{\max}(R)$ represented by px: $D_R \longrightarrow R_R$.

Claim. $[x]=\sum[L_g\bar{p}\bar{x}]$

$\sum[L_g\bar{p}\bar{x}]$ is represented by the S-homomorphism; $DS_S \longrightarrow S_S$: $y \longmapsto \sum g(\bar{p}\bar{x}(y))$. Now for d in D and h in G, $x(dh)=x(d)h=\sum[(px)(d)]^{g^{-1}}gh$ and $\sum g(\bar{p}\bar{x}(dh))=\sum g[(px)(d)h]=(\sum g(px)(d))h=\sum[(px)(d)]^{g^{-1}}gh$. Hence $[x]=\sum[L_g\bar{p}\bar{x}]=f(\sum g[px])$ with $\sum g[px]$ in $Q_{\max}(R)G$. Hence f is onto. Resultly f is an isomorphism. Thus $Q_{\max}(R)G$ is isomorphic to $Q_{\max}(RG)$.

For a finite group G of automorphisms of a semiprime ring R, let $t=\sum g$ and $S=RG$. We note tR is a bi R^G-S module, the right action of S being $tr\sum r_g g=\sum t(rr_g)^g$. The left action of R^G is clear. Also R is a bi $S-R^G$ module, where $\sum r_g gr=\sum r_g r^{g^{-1}}$. In this case to consider the derived Morita context of the left S-module ${}_SR$, we note that $Hom({}_SR, {}_SR)=R^G$ and $Hom({}_SR, S)=tR$.

Lemma 3.6. The derived Morita context of ${}_SR$ is $\langle S, {}_SR_{R^G}, {}_{R^G}tR_S, R^G\rangle$, where pairings are $(\ ,\ )$: $tR\otimes_S R\to R^G$, $(ta, b)=tr(ab)$ and $[\ ,\ ]$: $R\otimes_{S^G}tR\longrightarrow S$: $[a, tb]=atb$.

Proof. Let p: $tR\longrightarrow Hom({}_SR, S)$ by $p(tr)=f$, where $f(r)=tr$. Then $Hom({}_SR, S)=tR$. And let $q: R^G\longrightarrow Hom({}_SR, {}_SR)$ by $q(r)=g$, where $g(r)=r$ for all r in R. Then $Hom({}_SR, {}_SR)=R^G$. Since $tc[a, tb]=tc(atb)=\sum gc(atb)=\sum(gca)tb=\sum(ca)^g tb=tr(ca)tb=(tc, a)tb$, and $[a, tb]c=(atb)c=a(tbc)=a(\sum gbc)=a(\sum(bc)^g)=atr(bc)=a(tb, c)$.

Therefore $\langle S, {}_SR_{R^G}, {}_{R^G}tR_S, R^G\rangle$ is the derived Morita context of ${}_SR$.

Now we prove the result of [8, Theorem 2] differently.

THEOREM 3.7. If the derived Morita context $\langle S, {}_SR_{R^G}, {}_{R^G}tR_S, R^G\rangle$ in nondegenerate, then $Q_{\max}(R)^G = Q_{\max}(R^G)$.

PROOF. Since $\langle S, {}_SR_{R^G}, {}_{R^G}tR_S, R^G\rangle$ is nondenerate, $\langle Q_{\max}(S), Q({}_SR_{R^G}), Q({}_{R^G}tR_S), Q_{\max}(R^G)\rangle$ is right normalized by Theorem 2.6. By Corollary 3.4 and Theorem 3.2, $Q_{\max}(S) = Q_{\max}(R) G$ and $Q({}_SR)$ is the left quotient module $Q_{\max}(R)$ over the ring $Q_{\max}(S)$. So we have $Q_{\max}(R^G) = Hom_{Q_{\max}(S)}(Q({}_SR), Q({}_SR))$ and $Hom_{Q_{\max}(R)G}(Q_{\max}(R), Q_{\max}(R)) = Q_{\max}(R)^G$.

As is elementary and well known, one can imbed a commutative integral domain in a field, being nothing else than the fractions created from the elements of the domain. O. Ore gave the appropriate conditions in order that this be possible for noncommutative rings without zero divisors. We shall give an account of this rather, more general situation below. But first a few definition are needed.

DEFINITION 3.8. An element in a ring R is said to be *regular* if it is neither a left nor right zero divisor in R.

DEFINITION 3.9. An extension ring $Q(R)$ of R is said to be *a left quotient ring for* R if:

1. every regular element in R is invertible in $Q(R)$.
2. every $x \in Q(R)$ is of the form $x = a^{-1}b$ where a, $b \in R$ and a is regular.

If $Q(R)$ is a left quotient ring of R we say that R is left order in $Q(R)$. In any ring R, for a nonempty subset

S of R let $l(S)=\{x\in R: xs=0$ for all $s\in S\}$. We call $l(S)$ the left annihilator of S and term a left ideal λ of R a left annihilator if $\lambda=l(S)$ for some appropriate S in R. We similarly define the right annihilator $r(S)$ of S and speak of a right ideal as a right annihilator.

DEFINITION 3.10. A ring R is said to be a *(left) Goldie ring* if:

1. R satisfies the ascending chain condition on left annihilators.
2. R contains no infinite direct sums of left ideals.

Clearly a left Noetherian ring, that is, one satisfying the ascending chain condition on left ideals is a Goldie ring. A ring R is said to be semiprime if it has no nonzero nilpotent ideals.

THEOREM 3.11 [Goldie]. Let R be a semiprime left Goldie ring. Then R has a left quotient ring $Q=Q(R)$ which is semisimple artinian.

There has been a great deal of interest in group of outer automorphism, i.e., automorphism g for which there does not exist a unit u such that $r^g=u^{-1}ru$ for all r in R. Let R be a semiprime ring with a finite group G of ring automorphisms of R. Let S denote the ring of quotients of R relative to the Gabriel filter which consists of all two sided ideals whose annihilator is 0. An automorphism g is called *X-outer* if $sr^g=rs$ for an s in S and for all r in R implies that $s=0$. The group is called *X-outer* if each g $(g\neq 1)$ in G is *X*-outer.

elements $a_1, a_2, \cdots, a_n$; $a_1^*, a_2^*, \cdots, a_n^*$ in R such that $\sum_{i=q}^{n} a_i a_i^{*g} = \delta_{1,g}$ for all g in G, where δ is the Kronecker delta.

S. U. Chase, D. K. Harrison and A. Rosenberg [2] have shown that R is a G-Galois extension of R^G if and only if R is a finitely generated projective R^G-module and the map j from RG to $End_{R^G}(R)$ defined by $j(xg)(y) = xy^g$ for x, y in R and g in G is a ring isomorphism.

THEOREM 3.14. If R is a von Neumann regular selfinjective ring and G is X-outer, then

(1) R is a G-Galois extension of R^G.

(2) R_{R^G} is injective.

PROOF: If I is an essential right ideal of R^G, then IR is an essential right ideal of R because G is X-outer and R is regular, self-injective. Hence R^G is nonsingular. Since $\langle S, R, Rt, R^G \rangle$ is nondegenerate, the nonsingularity of R^G implies those of $S = RG$, R_S and Rt_{RG}. So S is regular. Since R_S is finitely generated, R_S is projective and hence $tr(R) = R^G$. Therefore by Lemma 3.8, R_{RG} is finitely generated. On the other hand, since R^G is semi-prime, it is nonsingular and self-injective. Now since $\langle S, R, Rt, R^S \rangle$ is right normalized, R is a G-Galois extension of R^G.

(2) Since R_R is injective by Lemma 3.13, so is R_{RG}.

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COROLLARY 3.12 [S. Montgomery]. If R is semiprime ring and a finite group G of ring automorphism of R is X-outer. Then R is right Goldie if and only if R^G is right Goldie.

PROOF. If R is right Goldie, then $Q_{\max}(R)$ is semi-simple Artinian and G acts on $Q_{\max}(R)$ as X-outer. So G is completely outer on $Q_{\max}(R)$ and hence $Q_{\max}(R)^G = Q_{\max}(R^G)$ is semi-simple Artinian. Thus R^G is semiprime right Goldie.

Conversely, suppose R^G is right Goldie. Then since the context $\langle S, R, Rt, R^G\rangle$ is nondegenerate, R^G is semiprime and hence $Q_{\max}(R^G) = Q_{\max}(R)^G$ is semisimple Artinian. Now by Amitsur [1], Muller [7] and Theorem 3.2, the maximal quotient context $\langle Q_{\max}(R)G, Q_{\max}(R), Q_{\max}(R)t, Q_{\max}(R)^G\rangle$ is also nondegenerate and hence $Q_{\max}(R)t_{Q_{\max}(R)^G} = Q_{\max}(R)_{Q_{\max}(R)}$ has finite Goldie dimension. Hence $Q_{\max}(R)$ is finitely generated over $Q_{\max}(R)^G$ and so $Q_{\max}(R)$ is Artinian. From the nondegeneracy of the maximal quotient context, the semi-primitivity of $Q_{\max}(R)G$ follows from $Q_{\max}(R)^G$ and so $Q_{\max}(R)$ is semiprimitive. Hence R is right Goldie.

LEMMA 3.13. If R is a right rationally complete, semiprime ring and $\langle S, R, Rt, R^G\rangle$ is nondegenerate. Then

(1) R is right self-injective iff R_{R^G} in injective.

(2) If R^G is right self-injective then $tr(R) = R^G$.

(3) If $tr(R) = R^G$ then R_{R^G} is finitely generated.

PROOF. (1) Suppose R_{R^G} is injective. Then $Hom_{R^G}(R, R) = S$ is injective because R_{R^G} is torsion free with respect to torsion theory induced from the trace ideal $tr(R)$ of Rt_{R^G}. Therefore R_R is injective. In a similar fashion, since S is

torsion free with respect to the hereditary torsion theory induced from the trace ideal RtR of R_S, the self-injectivity of S implies that $Hom_S(R_S, S_S)=Rt_{R^G}=R_{R^G}$ is injective.

(2) Suppose R^G is right self-injective. Then by the same reason as in (1), $Hom_{R^G}(Rt_{R^G}, R^G{}_{R^G})=R_S$ is injective. Hence R_S is an S-direct summand of S_S. Therefore R_S is projective and hence $tr(R)=R^G$.

(3) Let $\mathscr{A}_S=\{A\in Mod\text{-}S|A\longrightarrow Hom_S(RtR, A)$: bijective$\}$ and $\mathscr{A}_{R^G}=\{B\in Mod\text{-}R^G|B\longrightarrow Hom_{R^G}(tr(R), B)$: bijective$\}$. Then $\mathscr{A}_S$ and $\mathscr{A}_{R^G}$ are quotient categories of $Mod\text{-}S$ and $Mod\text{-}R^G$, respectively corresponding to hereditary torsion theories induced by trace ideals RtR and $tr(R)$. By Muller [7, Theorem 3], two functors $Hom_S(R_S, -)$ and $Hom_{R^G}(Rt_{R^G}, -)$ induces equivalences between $\mathscr{A}_S$ and $\mathscr{A}_{R^G}$. Let Λ denote quotient functors with respect to hereditary torsion theories induced by trace ideals. Then since $Hom_S(R_S,S)=Hom_S(R_S,S)=Rt_{R^G}$ the lattice of $\mathscr{A}_S$-subobject of S and $\mathscr{A}_{R^G}$-subobject of Rt_{R^G} are lattice isomorphic. Now to prove (3); suppose $tr(R)=R^G$. Then every R^G-submodule of Rt_{R^G} is $\mathscr{A}_{R^G}$-subobject. Now assume to the contrary that Rt_{R^G} is not finitely generated. Then there is a totally ordered set $\{I_\alpha\}$ of proper R^G-submodules of Rt_{R^G} with $\bigcup\limits_\alpha I_\alpha=Rt$. Hence $\{Hom_{R^G}(R, I_\alpha)\}$ is a totally ordered set of right proper $\mathscr{A}_S$-subobject of S. Since $Hom_{R^G}(R, \bigcup\limits_\alpha I_\alpha)=\bigcup\limits_\alpha Hom_{R^G}(R, I)$, we have $\bigcup\limits_\alpha Hom_{R^G}(R, I_\alpha)=Hom_{R^G}(R, Rt)=S$. But this is impossible because S_S is finitely generated. Therefore $Rt_{R^G}=R_{R^G}$ is finitely generated.

A ring R is called *G-Galois extension* of R^G if there are

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SOME PROPERTIES OF PROJECTIVE REPRESENTATIONS OF SOME FINITE GROUPS

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The representation group G^* of metacyclic group $G=BH$, $H \Delta G$ is known. When $|B|$ is prime, representation G^* can be easily obtained. Using this fact, some properties of projective representation of G will be discussed.

THEOREM 1. Let $G=\langle x, y | x^n=1,\ y^p=1,\ y^{-1}xy=x^r\rangle$ where $(n, r)=1$, p is prime. Then the number of irreducible projective representation with degree 1 is $p(n, r-1)$ and the degree of the irreducible projectve representation of G is one or p.

PROOF. $H^2(G, K^*)\cong Z_q$ where $q=\dfrac{k(n, r-1)}{n}$, $k=\left(n, \dfrac{r^p-1}{r-1}\right)$. By [3] the representation group G^* of G is

$$\langle x, y, z | x^n=1,\ y^p=1,\ z^q=1,\ y^{-1}xy=zx^r,\ xz=zx,\ yz=zy\rangle.$$

Also $\langle x, z\rangle \triangleleft G^*$, $\langle y\rangle < G^*$ and $\langle x, z\rangle$ is abelian. So G^* is the semidirect product of $\langle y\rangle$ by $\langle x, z\rangle$.

Let T be a representation of $\langle x, z\rangle$. We define $T^a : h \longrightarrow T(a^{-1}a^h)$ for $h\in\langle x, z\rangle$ and $a\in\langle y\rangle$. Then $S_T=\{y^k\in\langle y\rangle | Ty^k \cong T\}$ is a subgroup of $\langle y\rangle$. Since $|\langle y\rangle|$ is prime, $S_T=\{e\}$

or $S_T=\langle y\rangle$. So by [4], $S_T=\langle y\rangle$ iff $(T\otimes\rho)$ has degree 1, where ρ is an irreducible representation of $\langle y\rangle$.

All the irreducible representation whose degree is 1 has the form $T\otimes\rho$. So we have

$$\begin{aligned}
Ty^k\cong T \text{ iff } & Ty^k(x^iz^j)=T(x^iz^j)\\
\text{iff } & Ty^k(x^iz^i)=T(y^{-k}x^iz^iy^k)\\
&=T(x^{ir^k}z^{j+(1+r+\cdots+r^{k-1})})\\
&=T(x)^{ir^k}T(z)^{j+(1+r+\cdots+r^{k-1})}\\
&=T(x)^iT(z)^j\\
\text{iff } & T(x)^{i(r^{k-1})}=1 \text{ and } T(z)^{1+r+\cdots+r^{k-1}}=1\\
\text{iff } & d_1\frac{1-r^k}{1-r}\equiv 0(\text{mod } n) \text{ and } d_2\ \frac{1-r^k}{1-r}\equiv 0(\text{mod } q)
\end{aligned}$$

where $T(x)=\xi_1{}^{d_1}$, $T(z)=\xi_2{}^{d^2}$.

(ξ_1,ξ_2 are n, q-th roots of 1, respectively.)

So

$$S_T=\langle y\rangle \text{ iff } d_1(1-r^k)\equiv 0(\text{mod } n) \text{ and } d_2\frac{1-r^k}{1-r}\equiv 0(\text{mod } q)$$

for all k, $0\leq k\leq p-1$

and

$$S_T=\langle y\rangle \text{ iff } d_1(1-r)\equiv 0(\text{mod } n),$$

because $(1+r,\ 1+r+r^2,\cdots,1+r+\cdots+r^{k-1})=1$. So such $\{d_1\}$ is $(n, r-1)$. Therefore $\{T\otimes\rho\}$ is $p(n,\ r-1)$.

Since S_T is $\{e\}$ or $\langle y\rangle$, the degree of irreducible representation G^* is 1 or p. So the degree of projective irreducible representation of G is 1 or p. So our proof is completed.

THEOREM 2. Let $G=\langle x, y|x^n=1=y^p, y^{-1}xy=x^r\rangle$ with $(r,n)=1$, p prime, and $1+r+\cdots+r^{p-1}\equiv 0(\mathrm{mod}\ n)$.

(1) Then $p=q$, $p|n$ and $H^2(G, K^*)=\{1, \{\alpha\}, \cdots, \{\alpha^{p-1}\}\}$.

(2) For each $\{\alpha^k\}$, there exists exactly n/p linearly inequivalent projective representations with factor set $\{\alpha^k\}$.

(3) In this case

$$T_{ki}(x)=\mathrm{diag}\{\xi^{k+i},\ \xi^{k(1+r)+ri}, \ldots, \xi^{k(1+r+\cdots+r^{p-1})+r^{p-1}i}\},$$

$$T_{ki}(y)=\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

$$T_{ki}(y^jx^l)=T_{ki}(y)^jT_{ki}(x)^l,$$

and

$$\alpha(y^jx^i,\ y^lx^m)=\xi^{(1+r+\cdots+r^{l-1})i},$$

where ξ is a primitive n-th root of unity.

PROOF. In this situation we have $k=\left(n,\ \frac{r^p-1}{r-1}\right)=n$, $q=k(n,r-1)/n=(n,\ r-1)=d$. Therefore $r^p-1=(r-1)(r^{p-1}+\cdots+1)\equiv 0(\mathrm{mod}\ n)$ and hence $0=1+r+\cdots r^{p-1}=1+1+\cdots+1\equiv p(\mathrm{mod}\ d)$. But p is prime so $d=1$ or $d=p$. So $d=q=1$, $H^2(G, C^*)=\{e\}$, $d=p=q$, and $p|n$ since $(n,\ r-1)=d=p=q$. Now by [1], T_{ki} is a projective irreducible representation of G wich degree p with the factor set $\{\alpha^k\}$. Also their equivalence can be found in [1]. So for each $\{\alpha^k\}$ we have n/p linearly inequivalent projective representation.

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NOTES ON THE PSEUDO-COMPLETE ALGEBRA

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1. Introduction

In [5], Rickart proved that, when F is a Hermitian functional on the Banach $*$-algebra A, in order for F to be representable, it is necessary and sufficient that

(i) F is bounded,

(ii) $|F(x)|^2 \leq \mu F(x^*x)$, $x \in A$

where μ is a positive real constant independent of x. In this note, conditions for a functional to be admissible on a locally convex $*$-algebra are defined and sufficient conditions for a functional F to be representable are also given in Theorem 4.2.

2. Preliminaries

DEFINITION 2.1. By a *locally convex algebra* A we shall mean an algebra A over the complex field C, equipped with a topology τ such that

(i) $(A;\tau)$ is a Hausdorff locally convex topological vector space,

(ii) multiplication is separately continuous.

A will be called a *locally convex $*$-algebra* if A has a continuous involution.

DEFINITION 2.2. Let A be a locally convex algebra. An element x of A is said to be *bounded* if, for some nonzero complex number λ, the set $\{(\lambda x)^n : n \in N\}$ is a bounded subset of A.

The set of all bounded elements of A will be denoted by A_0.

NOTATION. By B_1 we denote the collection of all subsets B of A such that

(i) B is convex and idempotent,

(ii) B is bounded and closed.

If $B \in B_1$, then $A(B)$ will denote the subalgebra of A generated by B, i.e., $A(B) = \{\lambda x : \lambda \in C \text{ and } x \in B\}$, and the equation $\|x\|_B = \inf\{\lambda > 0 : x \in \lambda B\}$ defines a norm which makes $A(B)$ a normed algebra.

DEFINITION 2.3. The locally convex algebra A is called *pseudo-complete* if each of the normed algebras $A(B)$ is a Banach algebra.

If A is a locally convex algebra and $x \in A$, we define the *radius of boundedness* of x by

$$\beta(x) = \inf [\lambda > 0 : \{(\lambda^{-1}x)^n : n \in N\} \text{ is bounded}]$$

with the usual convention that $\inf \phi = \infty$.

The following simple facts about $\beta(x)$ are obvious:

1°. $\beta(x) \geq 0$ and $\beta(\lambda x) = |\lambda| \beta(x)$ where $\lambda \in C$ and $0 \cdot \infty = 0$.

2°. $\beta(x) < \infty$ iff $x \in A_0$.

3°. In particular, if A is pseudo-complete, then $\beta(x)$ equals to the spectral radius of x [1].

DEFINITION 2.4. Let A be a locally convex $*$-algebra, and let F be a linear functional on A. If $F(x^*)=(F(x))^-$ for all x in A, then F will be called *Hermitian*. If $F(x^*x)\geq 0$ for all x in A, then F will be called a *positive functional*.

LEMMA 2.5. Let A be a pseudo-complete locally convex $*$-algebra and let x_0 be any element of A such that $\beta(x_0)<1$. Then there exists an element y_0 of A such that $2y_0-y_0^2=x_0$. In addition if x_0 is Hermitian, so is y_0.

PROOF. Consider the function f defined in terms of the binomial series as follows:

$$f(z)=-\sum_{n=1}^{\infty}\binom{1/2}{n}(-z)^n.$$

Then f is well-defined and $2f(z)-[f(z)]^2=z$ for all $|z|\leq 1$. Now consider the vector valued function $-\sum_{n=1}^{\infty}\binom{1/2}{n}(-x_0)^n$.

We show that this series converges. Let $\varepsilon>0$. Since $\beta(x_0)<1$, there exists a $B\in B_1$ by [1] such that $x_0\in A(B)$ and $||x_0||_B<1$. Since f converges for $|z|\leq 1$, there exists an n_0 such that for $p, q>n_0$

$$\left\| \sum_{n=p}^{q-1}\binom{1/2}{n}(-x_0)^n \right\|_B <\varepsilon.$$

Since $A(B)$ is complete, we have that vector valued series converges to an element y_0 of $A(B)$ such that $2y_0-y_0^2=x_0$.

THEOREM 2.6. Let A be a pseudo-complete locally convex $*$-algebra and let F be any positive functional on A. Then

$|F(u^*hu)|<\beta(h)F(u^*u)$ for all $u\in A$ and h Hermitian.

PROOF. By Lemma 2.5 and [5, Theorem 4.5.2], the above theorem is obvious.

Let F be a positive functional on A and define

$$L_F=\{x\in A:\ F(y^*x)=0 \text{ for all } y \text{ in } A\}.$$

Then L_F is a left ideal of A([3, p. 288]). Now we define $X_F=A/L_F$ and denote $x+L_F$ by $\bar{x}$.

DEFINITION 2.7. A positive linear functional F which satisfies the following conditions will be called *admissible*:

(1) $\sup\{F(x^*a^*ax)/F(x^*x): x\in A\}<\infty$ for all $a\in A$.

(2) For each $x\in A$, there is a $x_0\in A_0$ such that $\bar{x}=\bar{x}_0$.

COROLLARY 2.8. If A is a pseudo-complete locally convex $*$-algebra such that $A=A_0$, then any positive functional is admissible.

PROOF. By Theorem 2.6 and 2°,

$$\{F(x^*a^*ax): x\in A=A_0\}\leq\beta(a^*a)<\infty \text{ for all } a\in A.$$

Since $A=A_0$ for each $x\in A$, there exists a $x_0(=x)\in A_0$ such that $\bar{x}=\bar{x}_0$.

3. Topologically Cyclic Representation

Let A be a $*$-algebra over the complex field C and X a vector space over C. A $*$-homomorphism $A\to L(X)$ is called a $*$-representation of A on X, where $L(X)$ is an algebra of all linear transformations of X into itself.

LEMMA 3.1. Let A be a locally convex $*$-algebra and let F be an admissible positive functional on A. If a, $b \in A$, then $\overline{(a+b)}_0 = (\bar{a}_0 + \bar{b}_0)$.

THEOREM 3.2. Let F be an admissible positive Hermitian functional on the commutative locally convex $*$-algebra A. Then there exists a representation $a \to T_a$ of A on a Hilbert space H such that $(T_a)^* = T_{a^*}$ for all $a \in A_0$.

PROOF. Since A is commutative, L_F is a two-sided ideal and hence X_F is an algebra. Let $\bar{x} = x + L_F$ and define a scalar product in X_F by $(\bar{x}, \bar{y}) = F(y^*x)$, for $x, y \in A$. The completion of X_F with respect to the inner product will be called H, and then H is a Hilbert space.

Let $\bar{x}_0$ be a fixed element of X_F. Since F is admissible, we may assume that $x_0 \in A_0$. Let $\bar{z} \in H$ and assume that $\bar{z}_n \to \bar{z}$ with $\bar{z}_n \in X_F$.
Then

$$\begin{aligned} ||\bar{x}_0\bar{z}_n - \bar{x}_0\bar{z}_m||^2 &= (\bar{x}_0\bar{z}_n - \bar{x}_0\bar{z}_m,\ \bar{x}_0\bar{z}_n - \bar{x}_0\bar{z}_m) \\ &= F((x_0z_n - x_0z_m)^*(x_0z_n - x_0z_m)) \\ &= F((z_n - z_m)^*x_0^*x_0(z_n - z_m)) \end{aligned}$$

and

$$||\bar{z}_n - \bar{z}_m||^2 = F((z_n - z_m)^*(z_n - z_m)).$$

Since F is admissible,

$$||\bar{x}_0\bar{z}_n - \bar{x}_0\bar{z}_m||^2 \leq M||z_n - z_m||^2 \text{ with } M > 0.$$

Thus $\{\bar{x}_0\bar{z}_n\}$ is a Cauchy sequence with respect to the inner product norm, and hence the sequence converges to an element $\bar{y}$ of H. Similarly we can show that if $\bar{w}_n \to \bar{z}$ with

respect to the inner product norm, then $\{\bar{x}_0\bar{w}_n\}$ converges to $\bar{y}$. Now we define the mapping $a\to T_a$ of A on H by

$$T_a\bar{x}=\bar{a}_0\bar{x},\quad \bar{x}\in H \text{ where } \bar{a}_0=\bar{a}.$$

Then, if $a,\ b\in A$,

$$\begin{aligned} T_{ab}\bar{x} &= (ab)^{-}{}_0\bar{x}=(ab)^{-}\bar{x}=\bar{a}b\bar{x}=\bar{a}_0\bar{b}_0\bar{x} \\ &= (a_0(b_0x))^{-}=T_a(b_0x)^{-} \\ &= T_aT_b\bar{x} \quad \text{for all } \bar{x}\in H. \end{aligned}$$

Similarly $T_{a+b}=T_a+T_b$ and $T_{\lambda a}=\lambda T_a$ for all $\lambda\in C$. Thus $a\to T_a$ defines a representation of A on H.

Consider the restriction of the representation to A_0. Let $a\in A_0$. Since F is admissible, we have

$$\begin{aligned} \|T_a(\bar{x})\|^2 &= \|\bar{a}\bar{x}\|^2=(\bar{a}\bar{x},\bar{a}\bar{x}) \\ &= F(x^*a^*ax) \\ &\le M\|\bar{x}\|^2 \text{ for some } M>0,\quad \bar{x}\in X_F. \end{aligned}$$

Hence T_a is a continuous mapping on X_F. Since X_F is dense in H, T_a can be uniquely extended to a continuous mapping $\hat{T}_a$ on H. However if $\bar{x}\in H-X_F$, let $\{\bar{x}_n\}$ be a subset of X_F such that $\bar{x}_n\to x$. Then

$$\begin{aligned} \hat{T}_a(\bar{x}) &= \lim\ \hat{T}_a(\overline{x_n})=\lim\ T_a(\overline{x_n})=\lim\ \bar{a}\overline{x_n} \\ &= \bar{a}\bar{x}=T_a(\bar{x}). \end{aligned}$$

Thus $\hat{T}_a=T_a$ and T_a is a continuous function on H for $a\in A_0$. Since T_a is continuous, we can show that $(T_a)^*=T_{a^*}$ by proving that $(T_a)^*(x)=T_{a^*}(x)$ for all $x\in X_F$.

Let $\bar{x}$ and $\bar{y}$ be elements of X_F, then

$$(T_a\bar{x}, \bar{y}) = F(y^*ax) = F((y^*a)x)$$
$$= (x, (\bar{a}^*)\bar{y}) = (x, T_{a^*}\bar{y}).$$

Thus for $a \in A_0$, we have $(T_a)^* = T_{a^*}$.

COROLLARY 3.3. If A_0 is also an algebra e.g., the product of bounded sets of A is bounded, then the restriction of the above representation to A_0 is a $*$-representation of A_0 on H.

Let X be a vector space over C and let K be a subalgebra of $L(X)$. Let z be a fixed vector in X and let $X_z = \{T(z) : T \in K\}$. Then X_z is an invariant subspace of X with respect to K. If there exists an element z of a normed space X such that $X_z = X$, then K is said to be *topologically cyclic* and the vector z is called a *topologically cyclic vector.* A representation $x \to T_x$ of A on X is said to be *topologically cyclic* if, when $K = \{T_x : x \in A\}$, there is a vector z in X such that $X_z = X$.

With these definitions we state the following corollary to Theorem 3.2.

COROLLARY 3.4. Let A be a commutative locally convex $*$-algebra with identity. Let F be an admissible positive Hermitian functional on A. Then the representation obtained above is topological cyclic with a cyclic vector h_0 such that $F(x) = (T_x h_0, h_0)$, $x \in A$.

PROOF. Let $h_0 = \bar{1} = 1 + X_F$. Then by definition $T_x h_0 = \bar{x}_0$, so that the set $\{T_x h_0 : x \in A\} = X_F$ and hence is dense in H. Thus h_0 is a topologically cyclic vector. Now let $x \in A$, then there exists $x_0 \in A_0$ such that $\bar{x} = \bar{x}_0$. Thus

$$F(1^*(x-x_0))=F(x-x_0)=F(x)-F(x_0).$$

By the way, $F(1^*(x-x_0))=((x-x_0)^-,\ \bar{1})$

$$=(\bar{x},\bar{1})-(\bar{x},\bar{1})=0.$$

Consequently $F(x)=F(x_0)$. Therefore $(T_x h_0, h_0)=(\bar{x}_0 h_0, h_0)$ $=(\bar{x}_0\bar{1}, 1)=F(x_0)=F(x)$ for all $x\in A$.

4. Representable Functional

Let F be a linear functional on the locally convex $*$-algebra A and let $a \to T_a$ be a representation of A on a Hilbert space H such that the restriction of the representation to A_0 is a $*$-representation of A_0 on H. Then F is said to be *representable by* $a \to T_a$ provided there exists a topologically cyclic vector $h_0\in H$ such that

$$F(a)=(T_a h_0, h_0) \text{ for all } a\in A.$$

Let $a \to T_a$ be a representation of A on H and let

$$M=\{h\in H : T_a h=0 \text{ for all } a\in A\}.$$

If $M=\{0\}$, we say that the representation is *essential*.

LEMMA 4.1. If the representation $a \to T_a$ is essential, then each of the subspaces $H_h=\{T_a h: a\in A\}$ is cyclic with h as a cyclic vector.

PROOF. [5, p. 206].

THEOREM 4.2. Let F be a Hermitian functional on the pseudo-complete commutative locally convex $*$-algebra A. Then in order for F to be representable, it is sufficient that

(1) for each $x \in A$, there is a $x_0 \in A_0$ such that $\bar{x} = \bar{x}_0$,

(2) $|F(x)|^2 \leq \mu F(x^*x)$, $x \in A$,

where μ is a positive real constant independent of x.

PROOF. Assume that F satisfies the conditions and denote by A_1 the pseudo-complete locally convex $*$-algebra obtained by adjoining the identity element to A. Extend the functional F to A_1 by the definition,

$$F(x+\alpha) = F(x) + \mu\alpha \text{ for } x \in A \text{ and } \alpha \text{ a scalar.}$$

Then

$$\begin{aligned} F((x+\alpha)^*(x+\alpha)) &= F((x^*+\bar{\alpha})(x+\alpha)) \\ &= F(x^*x + x^*\alpha + \bar{\alpha}x + \bar{\alpha}\alpha) \\ &\geq F(x^*x) - 2|\alpha||F(x)| + \mu|\alpha|^2 \\ &\geq F(x^*x) - 2|\alpha|\mu^{\frac{1}{2}} F(x^*x)^{\frac{1}{2}} + \mu|\alpha|^2 \\ &= (F(x^*x)^{\frac{1}{2}} - |\alpha|\mu^{\frac{1}{2}})^2. \end{aligned}$$

Thus F is a positive linear functional on A_1 and Theorem 2.6 guarantees that the first condition of admissibility is satisfied on A_1. To show that the second condition is satisfied, let $x+\alpha \in A_1$. Then by hypothesis there exists $x_0 \in A_0$ such that $\bar{x}_0 = \bar{x}$. Consider $x_0 + \alpha$. Then since $\bar{x}_0 = \bar{x}$ and $(x - x_0) \in L_F$,

$$\begin{aligned} &|F[(y+\beta)^*((x_0+\alpha)-(x+\alpha))]|^2 \\ &= |F[(y+\beta)^*(x_0-x)]|^2 \\ &= |F(y^*(x-x_0)) + F(\bar{\beta}(x_0-x))|^2 \\ &= |\bar{\beta}F(x_0-x)|^2 \\ &\leq |\beta|^2 F[(x_0-x)^*(x_0-x)] = 0. \end{aligned}$$

Consequently $(x_0+\alpha)^- = (x+\alpha)_0^-$.

Therefore F is an admissible positive Hermitian func-

tional on A_1. Hence by Corollary 3.4 there exists a representation $x \to T_x$ of A_1 on H defined by $T_{(a+\alpha)}x = (a+\alpha)_0 \overline{x}$ and such that

$$F(a+\alpha) = (T_{a+\alpha}h_0, h_0) \text{ for some } h_0 \in H.$$

Now let $N = \{h \in H : T_a h = \theta \text{ for all } a \in A\}$.

Consider the restriction of $a \to T_a$ to the space $N^{\perp}$, where

$$N^{\perp} = \{h \in H : (h, n) = 0 \text{ for all } n \in N\}.$$

Since $\{h \in N^{\perp} : T_a h = \theta \text{ for all } a \in A\} = \{0\}$, the restriction is essential.

Let $h_0 = h_0' + h_0''$ where $h_0' \in N^{\perp}$ and $h_0'' \in N$. Then for all $a \in A$ we have

$$\begin{aligned} F(a) &= (T_a h_0, h_0) = (T_a(h_0' + h_0''),\ h_0' + h_0'') \\ &= (T_a h_0',\ h_0' + h_0'') = (h_0',\ T_a^*(h_0' + h_0'')) \\ &= (h_0',\ T_a^* h_0') = (T_a h_0', h_0'). \end{aligned}$$

Thus there exists $h_0' \in N^{\perp}$ such that $F(a) = (T_a h_0', h_0')$ for all $a \in A$. Let $H_0 = \{T_a h_0' : a \in A\}$. Then, since the restriction of the representation to $N^{\perp}$ is essential, by Lemma 4.1 H_0 is cyclic with h_0 as a cyclic vector.

COROLLARY 4.3. If A has an identity element, then every positive functional which implies condition (1) is representable.

PROOF. If A has an identity element, then by the Cauchy-Schwarz inequality, we have

$$|F(x)|^2 \leq F(1)F(x^*x)$$

for any positive functional F. Thus, condition (2) is au-

tomatically satisfied.

COROLLARY 4.4. Let F be an admissible positive Hermitian functional on the pseudo-complete commutative locally convex $*$-algebra A. Then there exists a $*$-representation of A_0 on a Hilbert space H.

PROOF. If A is commutative and pseudo-complete, then A_0 is an subalgebra of A [1]. Therefore by Theorem 3.2 and Corollary 3.3, the proof is obvious.

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A NOTE ON FUZZY TOPOLOGY, FUZZY GROUPS AND FUZZY TOPOLOGICAL GROUPS

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1. Introduction

Zadeh's introduction [17] of the notion of a fuzzy set in a universe could generalize and extend main concepts and structures of the presentday mathematics into the framework of fuzzy sets. Goguen [6] has studied and generalized the work of Zadeh. The most generalization was the consideration of order structures beyond the unit closed interval. The concept of a fuzzy topological space and some of its basic notions have been studied by Chang [4] as one of applications of concepts of a fuzzy set. In the development of a parallel theory based on fuzzy sets, many interesting phenomena have been observed. For example, the concept of compact fuzzy topological spaces introduced in the literature by Chang holds only for finite products. The next compactness results by Goguen are an Alexander Subbase Theorem and a Tychonov Theorem for finite products and he was the first to point out a deficiency in Chang's compactness. Weiss [15] and Lowen [8] introduced new definitions of compactness in a fuzzy topological space which is available for the infinite products. However, the definition of a fuzzy topological space by them has been pointed

out that it has the great deficiency that its definition is not the generalization of an ordinary topological space.

In [13] Resenfield has used the notion of a fuzzy set to develop theory of fuzzy groups. In [1] authors have point out a deficiency in Rosenfield's definition of fuzzy groups. They have used a different structure to define a fuzzy group. The structure is one of stronger conditions than Rosenfield's. Foster [5] has introduced a fuzzy topological group by use of definitions of Lowen's and Rosenfield's. In [11] authors defined a fuzzy topological group by use of the concept of Q-neighborhood introduced in [12] and showed that thier definition and Foster's definition are equivalent under some condition.

Let X be an ordinary nonempty set which we will call the *universe*. A fuzzy set A in X is a function on X into the closed unit interval $[0,1]$, assigning each x in X to its grade of membership $A(x)$ in A. The grade of membership function is often called a *generalized characteristic function.* Fuzzy set operations; inclusion, union, intersection, generalized union and intersection, complement; are defined by use of $\leq$, max, min, sup and inf, $1-$, similarly to the corresponding notions in ordinary set operations, respectively. It is one of important problems that it was shown that in Zadeh's structure of fuzzy set theories the class of generalized characteristic functions is a distributive but noncomplementary lattice and it is just a Brouwerian lattice. Roughly to speak, $A \cap A' = \phi$ does not hold in the fuzzy structure, where A' denotes the complement of A. In fuzzy structure there are problems left, for example, a fuzzy point, compactness in fuzzy topological spaces, fuzzy

neighborhoods problems.

In this paper we will use definitions which, we think, are most suitable in the presentday publications. We can find definition without mentions in the available references.

2. Fuzzy points, fuzzy topologies and fuzzy neighborhoods.

How to define a fuzzy point reasonably in a fuzzy set is one of fundamental problems in fuzzy structures. In [12, 16, 17] a fuzzy point was defined in different ways. We will follow the definition in [12, 17], named a fuzzy point instead of a fuzzy singleton in [6].

DEFINITION 2.1. A fuzzy set in X is called a *fuzzy point* if it takes the value 0 for all point y in X except one, say x in X. If its value at x is k in $(0, 1]$, then we will denote the fuzzy point by a lowercase letter $x(k)$.

DEFINITION 2.2. Let $x(k)$ be a fuzzy point and A a fuzzy set in a universe X. Then $x(k)$ is said to *be in* A or A *contains* $x(k)$, denoted by $x(k) \in A$ (or simply $x(k)$ in A), if $k \leq A(x)$ all x in X.

Evidently every fuzzy set A can be expressed as a union of all fuzzy points which belong to A. As we will know later on, the concept of fuzzy points is very important for the construction of fuzzy neighborhood in fuzzy topological spaces. When a mapping between universes is defined, the inverse image and image of fuzzy sets in them were defined almost similarly to those in ordinary sets.

DEFINITION 2.3. Let $f: X \to Y$ be a mapping of universe X

into universe Y, and A and B fuzzy sets in X and Y, respectively. Then the *inverse image* of B, $f^{-1}(B)$, is the fuzzy set in X with membership given by $f^{-1}(B)(x)=B(f(x))$ for all x in X and the *image* of A, $f(A)$, is the fuzzy set in X with membership given by

$$f(A)(y)=\sup_{z\in f^{-1}(y)} A(z), \text{ if } f^{-1}(y)\neq\phi$$
$$= 0 \qquad , \text{otherwise}$$

for all y in Y, where $f^{-1}(y)=\{x|f(x)=y\}$.

DEFINITION 2.4. Let A and B be fuzzy sets in universes X and Y, respectively. The *fuzzy product* $A\times B$ of A and B is defined as the fuzzy set in the usual set product $X\times Y$ with the membership given by $A\times B(x, y)=\min\ (A(x), B(x))$ for all (x, y) in $X\times Y$.

PROPOSITION 2.5. Let p_i be the projection of $X_1\times X_2$ into X_i, for $i=1, 2$, and $A=A_1\times A_2$ a fuzzy product in $X_1\times X_2$. Then $p_i(A)\subset A_i$, for each $i=1, 2$.

PROOF. If $i=1$, $p_1(A)(x_1)=\sup_{(z_1, z_2)\in p_1^{-1}(x_1)} A(z_1,z_2)$

$$=\sup_{(z_1, z_2)\in p_1^{-1}(x_1)} \min(A_1(z_1), A_2(z_2))$$

$$=\min(A_1(x_1),\ {}_{z_2}\sup_{X_2}A_2(z_2))$$

for all x_1 in X_1. Similarly we can prove the case of $i=2$.

Let X be a universe. Then a family $\mathcal{T}$ of fuzzy sets in X is called a *fuzzy topology* on X if (i) $\phi, X\in\mathcal{T}$, where ϕ is a fuzzy empty set (ii) If $A, B\in\mathcal{T}$, then $A\cap B\in\mathcal{T}$ (iii)

If $A_i \in \mathcal{T}$ for all $i \in I$, then $\bigcup_{i \in I} A_i \in \mathcal{T}$. The pair $(X, \mathcal{T})$ is called a *fuzzy topological space*(for short, fts) and the members of $\mathcal{T}$ are called *$\mathcal{T}$-open fuzzy set*. The complement of a $\mathcal{T}$-open fuzzy set is called a *$\mathcal{T}$-closed fuzzy set*. We will drop $\mathcal{T}$ without confusions. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are fuzzy topologies on a universe X and $\mathcal{T}_1 \subset \mathcal{T}_2$, then we say that $\mathcal{T}_2$ is *finer* than $\mathcal{T}_1$ or $\mathcal{T}_1$ is *coarser* than $\mathcal{T}_2$. A *base* and a *subbase* of an fts were defined as the similar way in an ordinary topological space and the *interior* of a fuzzy set is defined as the largest open fuzzy set contained in the fuzzy set and the *closure* of a fuzzy set is defined as the smallest closed fuzzy set containing the fuzzy set. The properties of the interior and the closure are like those in the usual topological spaces.

The neighborhood of a fuzzy point in an fts has been defined in different manners [4, 6, 8, 12, 15, 16]. A fuzzy set N in an fts $(X, \mathcal{T})$ is called a *neighborhood*(for short, nbd) of fuzzy point $x(k)$ if there is an O in $\mathcal{T}$ such that $x(k) \in O \subset A$. In [12], corresponding to this, authors have defined a more reasonable definition by use of a new concept. We will use it.

DEFINITION 2.6. A fuzzy point $x(k)$ is said to be *quasi-coincident* with a fuzzy set A, denoted by $x(k)\ q\ A$, if $k > A'(x)$ or $k + A(x) > 1$. The quasi-coincident with two fuzzy sets A and B, denoted by $A\ q\ B$, means that there exists x in X such that $A(x) > B'(x)$, or $A(x) + B(x) > 1$.

DEFINITION 2.7. A fuzzy set N in an fts $(X, \mathcal{T})$ is called a *Q-nbd* of $x(k)$ if there is an O in $\mathcal{T}$ such that $x(k)$ $q\ O \subset N$; a nbd N is said to be *open* iff N is open.

It was shown in [10] that $A \subset B$ iff A and B' are not quasi-coincident; $x(k)$ is in a fuzzy set A iff $x(k)$ is not quasi-coincident with A'. From the fact, the substitute for the fact that A and A' do not intersect in general topology is the fact that A and A' are not quasi-coincident in fuzzy topology. This means much suitability for definition, while in Zadeh's theory the class of generalized characteristic fuctions is just a Brouwerian lattice.

Let $(X, \mathcal{T})$ be an fts and A a fuzzy set in X. It is easy to prove that the family $\mathcal{T}_A = \{A \cap U | U \in \mathcal{T}\}$ is a fuzzy topology on A. Thus we say that the pair $(A, \mathcal{T}_A)$ is called a *fuzzy subspace* of $(X, \mathcal{T})$. A mapping f of an fts $(X, \mathcal{T})$ into an fts $(Y, \mathcal{U})$ is said to be *fuzzy continuous* (for short, F-continuous) if, for each B in $\mathcal{U}$, $f^{-1}(B)$ is in $\mathcal{T}$. We will denote a mapping f an fts $(X, \mathcal{T})$ into an fts $(Y, \mathcal{U})$ by $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$.

PROROSITION 2.8. Let $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a mapping. Then the following are equivalent:

(1) f is F-continuous.

(2) For each $\mathcal{U}$-open fuzzy set V, $f^{-1}(V)$ is $\mathcal{T}$-open fuzzy set.

(3) For any nbd V of $f(x(k))$, there exists a nbd U of $x(k)$ such that $f(U) \subset V$.

(4) For each fuzzy point $x(k)$ in X and each Q-nbd V of $f(x(k))$, there exists a Q-nbd U of $x(k)$ such that $f(U) \subset V$.

PROOF. We will prove (1)$\Longleftrightarrow$(4) and leave the remainder for readers.

Let $f(x)=y$. Then $f(x(k))=y(k)$ from definitions. Since V is a Q-nbd of $f(x(k))$, there is W in $\mathcal{U}$ such that $W \subset V$ and $W(y)+k>1$. Let $f^{-1}(W)=U$. Then U is in $\mathcal{T}$ and $U(x)+k=W(y)+k>1$. Thus U is a Q-nbd of $x(k)$ and $f(U)=ff^{-1}(W) \subset V$. The converse is obvious.

REMARK 2.9. It is easy to prove that the restriction of a mapping f of $(X, \mathcal{T})$ to $(Y, \mathcal{U})$ and the composition $g \circ f$ of f and g is F-continuous if f and g are F-continuous and we can get some theorem for complete condition to be F-continuous by means of interior, closure and so on.

DEFINITION 2.10. Let $(A, \mathcal{T}_A)$ and $(B, \mathcal{U}_B)$ be fuzzy subspaces of fts's $(X, \mathcal{T})$ and $(Y, \mathcal{U})$, respectively. Then a mapping f: $(A, \mathcal{T}_A) \to (B, \mathcal{U}_B)$ is said to be *relatively fuzzy continuous* (for short, RF-continuous) if, for each W in $\mathcal{U}_B$, $f^{-1}(W) \cap A$ is in $\mathcal{T}_A$.

PROPOSITION 2.11. Let f: $(X, \mathcal{T}) \to (Y, \mathcal{U})$ be fuzzy continuous and $(A, \mathcal{T}_A)$, $(B, \mathcal{T}_B)$ fuzzy subspaces, respectively. If $f(A) \subset B$, then $f:(A, \mathcal{T}_A) \to (B, \mathcal{U}_B)$ is RF-continuous.

PROOF. To apply Proposition 2.8, let $a \in A$ and N a Q-nbd of $f(a)$ and $f(a)=b$. Then $f(a(k))=b(k)$. Since N is a Q-nbd of $f(a(k))$, there is a V in $\mathcal{U}_B$ such that $b(k)$ $qV \subset N$, that is, $V \subset N$, $V(y)+k>1$. Since V is in $\mathcal{U}_B$, there is an $O \in \mathcal{U}$ such that $V=B \cap O$. Thus $f^{-1}(V)=$ $f^{-1}(B \cup O)=A \cup f^{-1}(O)$. So we have $f^{-1}(V) \in \mathcal{T}_A$ because $f^{-1}(O) \in \mathcal{T}$ from the F-continuity of f. Let $U=f^{-1}(V)$. Then U is a Q-nbd of $a(k)$ such that $f(U) \subset V$.

DEFINITION 2.12. Let $\{(X_i, \mathcal{T}_i) | i \in I\}$ be a family of fts's

(2) $T(a,b)\leq T(c,d)$ whenever $a\leq c, b\leq d$ (monotonicity)

(3) $T(a,b)=T(b,a)$ (symmetric)

(4) $T(T(a,b),c)=T(a,T(b,c))$ (associativity)

EXAMPLE 3.2. T_W is defined by the boundary conditions and $T_W(a,b)=0$ for each (a,b) in $[0,1)\times[0,1)$, $\min(a,b)$, $T_m=\max(0,a+b-1)$ and $\mathrm{Prod}(a,b)=ab$ are t-norms.

REMARK 3.3. The t-norms in Example 3.2 hold obviously;

(1) $T_W(a,b)\leq T_m(a,b)\leq \mathrm{Prod}(a,b)\leq \min(a,b)$

(2) For any t-norm T, $T_W(a,b)\leq T(a,b)\leq \min(a,b)$

DEFINITION 3.4. Let X be a universal group and G a fuzzy set in X. Then G is called a *fuzzy group* in X if, for each x,y in X, (i) $G(xy)\geq T(G(x),G(y))$ (ii) $G(x^{-1})\geq G(x)$, where T is a t-norm defined.

PROPOSITION 3.5. G is a fuzzy group in X iff, for every x,y in X, $G(xy^{-1})\geq T(G(x),G(y))$.

PROOF. Let G be a fuzzy group in X. Then $G(xy^{-1})\geq T(G(x),\ G(y^{-1}))\geq T(G(x),G(y))$ because $G(x^{-1})\geq G(x)$ for all x in X and from the monotonicity of T. The converse follows from [13, 5.6] because we can replace min by T from Remark 3.3 and is thus omitted.

PROPOSITION 3.6. Let f be a homomorphism of group X into group Y in usual sense G a fuzzy group in Y. Then the inverse image $f^{-1}(G)$ of G is a fuzzy group in X.

PROOF. For all x,y in X, applying Proposition 3.5, $f^{-1}(G)(xy^{-1})=G(f(xy^{-1}))=G(f(x)(f(y))^{-1})\geq T(G(f(x)),$ $G(f(y)^{-1})\geq T(G(f(x)),G(f(y)))$.

REMARK 3.7. Similarly, it can be shown without much difficulty that the image $f(G)$ of a fuzzy group G under the homomorphism f(in usual sense) is a fuzzy group.

PROPOSITION 3.8. Let G be a fuzzy group in a group X and e the identity in X. Then $G(x^{-1})=G(x)$ and $G(x)\leq G(e)$.

PROOF. $G(x)=G((x^{-1})^{-1})\geq G(x^{-1})\geq G(x)$ for all x in X. The remainders can be shown similary to Proposition 3.5 and are thus omitted.

Kaufmann [7] has introduced the concept of ordinary subset of level t of a fuzzy set to decompose a fuzzy set into a ordinary set. Lat A be a fuzzy set of X. Then the ordinary set $A_t=\{x\in X|A(x)\geq t\}$ is called a *level subset* of fuzzy set A. He has shown that every fuzzy set can be decomposed as products of ordinary susets (i.e., the level subsets) and a number in [0, 1]. Thus some questions will be arised; what a level subset of a fuzzy group of a universal group will be a subgroup of the group? One of the answer is, so-called, level subgroup, the level subset $A_t=\{x\in X|t\leq A(e),\ t\in[0,1]$ and e is the identity in $X\}$. It can prove that the A_t is a subgroup in X; The number of such level subgroups in X may depend on t. Since $t\in[0,1]$, there can be an infinite number of level subgroups in X although X is finite. However it means a contradiction because the number of all subsets of a finite group must be finite. Thus we have a question: when level subgroups of a fuzzy groups are equal each other?

Let G be a fuzzy group X and $G_e=\{x\in X|G(x)=G(e)\}$.

and $X = \prod_{i \in I} X_i$ the usual set product. Then $(X, \mathcal{T})$ is called the *product fts* if $\mathcal{T}$ is the coarsest fuzzy topology on X such the projection p_i of X onto X_i is fuzzy continuous for each $i \in I$. The fuzzy topology $\mathcal{T}$ is called the *product fuzzy* on X.

REMARK 2.13. Since the concept of the product fts is almost similar to that of the product space in the usual sense. Thus it can be shown that the product fuzzy topology $\mathcal{T}$ on X has the fuzzy set of the form $p^{-1}(U_i)$ as a subbase where U_i is in $\mathcal{T}_i$, $i \in I$. Therefore, the base for $\mathcal{T}$ is the form of finite intersection of $\{p_i^{-1}(U_i) \mid U_i \in \mathcal{T}_i\}$.

PROPOSITION 2.14. Let $\{(X_i, \mathcal{T}_i)\}$, $i \in I$, be a family of fts's, $(X, \mathcal{T})$ the product fts and $f: (Y, \mathcal{U}) \to (X, \mathcal{T})$ a mapping. Then f is F-continuous iff $p_i \circ f$ is F-continuous for each i.

PROOF. Let $B \in \mathcal{T}_i$. Then $(p_i \circ f)^{-1}(B) = (f^{-1} \circ p_i^{-1})(B)$ is in $\mathcal{U}$. Hence $\{(f^{-1}[p_i^{-1}(B)]) \mid i \in I\}$ is a family of $\mathcal{U}$-open fuzzy set in Y. Since f^{-1} preserves union and intersection in fuzzy sets as well as in ordinary sets, f is F-continuous. The converse is trivial.

PROPOSITION 2.15. Let $(X, \mathcal{T})$ be the product fts of $\{(X_i, \mathcal{T}_i) \mid i = 1, 2, \cdots, n\}$. Let each A_i be fuzzy set in X_i and A a product fuzzy set in X. Let B be a fuzzy set in a fts $(Y, \mathcal{U})$ and $f: (B, \mathcal{U}_B) \to (A, \mathcal{T}_A)$ a mapping. Then f is RF-continuous iff $p_i \circ f$ is RF-continuous for each $i \in I$.

PROOF. Apply Propositions 2.5, 2.11 and Remark 2.13.

PROPOSITION 2.16. Let A and B be fuzzy sets and C the

product fuzzy set of fts's $(X, \mathcal{T})$, $(Y, \mathcal{U})$ and $(Z, \mathcal{V})$ the product fts, respectively. Then for each $a \in X$ such that $A(a) \geq B(y)$ for all y in Y, the mapping $i: y \to (a, y)$ of $(B, \mathcal{U}_B)$ into $(C, \mathcal{V}_C)$ is RF-continuous.

PROOF. We have $i(B) \subset C$ from the membership function of $i(B)$ and the concept of product fuzzy set. It is shown that the identity and constant mappings are F-continuous. From this and the F-continuity of composition, we apply Proposition 2.14. Let $i_1: y \to a$ and $i_2: y \to y$ be mappings. Then $i = i_1 \circ i_2$. Using Proposition, the proof is complete.

3. Fuzzy groups and Fuzzy topological groups.

Rosenfield [13] has defined a fuzzy groups to extend and to generalize the notion of groups structures; let X be a universal group and G a fuzzy set in X with the grade of membership $G(x)$ for all x in X. Then G is called a *fuzzy group* in X if, for every x, y in X, (i) $G(xy) \geq \min(G(x), G(y))$ (ii) $G(x^{-1}) \geq G(x)$. In [1] authors pointed out a deficiency in it and gave examples which are groups in usual sense, but not fuzzy groups in sense of Rosenfield. This means that a fuzzy group may be not a generalization of a groups. To get rid of the default, they have used different operator, so called t-norm, to define a fuzzy group.

DEFINITION 3.1. A *t-norm* is a function $T: [0,1] \times [0,1] \to [0,1]$ satisfying; for each a, b, c, d in $[0,1]$,

(1) $T(0,0) = 0$, $T(a,1) = a = T(1,a)$ (boundary conditions)

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Then G_e is one of level subgroups in X and for $a \in X$, let r_a: $x \rightarrow xa$ and l_a: $x \rightarrow ax$ denote, respectively, right and left translations of X into itself.

PROPOSITION 3.9. Let G be a fuzzy group in a group X. Then for all $a \in G_e$, $r_a(G) = l_a(G) = G$.

PROOF: Let $a \in G_e$. Then $a^{-1} \in G_e$ since G_e is a subgroup. Since $G(e) = 1$ and $T(k, 1) = k$, $l_a(G(x)) = G(xa^{-1}) \geq T(G(x), G(e)) = G(x) = G(xa^{-1}a) \geq T(G(xa^{-1}), G(e)) = G(xa^{-1}) = l_a(G(x))$ for all x in X. The proof for r_a is similar to this.

We will study properties of fuzzy topological groups(for short, ftg) from now on. Foster [5] has defined an ftg by means of the fts in sense of Lowen and the fuzzy group in sense of Rosenfield. In [11] authors have defined it by use of Q-nbd of fuzzy points and showed that their concept is equivalent to Foster's. We will apply the definitions in this note to define an ftg.

Let X be a universal group and A, B fuzzy sets in X. We define AB and A^{-1} by the respective formulas; $AB(x) = \sup_{yz=x} \min(A(y), A(z))$ and $A^{-1}(x) = A(x^{-1})$ for each x in X.

DEFINITION 3.10. Let X be a universal group and $(X, \mathcal{T})$ an fts. Let G be a fuzzy group and $(G, \mathcal{T}_G)$ a fuzzy subspace of $(X, \mathcal{T})$. Then G is called an *ftg* if

(i) The mapping g: $(x, y) \rightarrow xy$ of $(G, \mathcal{T}_G) \times (G, \mathcal{T}_G)$ into $(G, \mathcal{T}_G)$ is RF-continuous.

(ii) The mapping h: $x \rightarrow x^{-1}$ of $(G, \mathcal{T}_G)$ onto itself is RF-

continuous.

EXAMPLE 3.11. Let $X=(R,+)$ be the usual topological group. Let $\mathcal{L}$ be a family of all lower semicontinuous functions of X into $[0,1]$. Then $(X,\mathcal{L})$ is ftg because $\mathcal{L}$ is a fuzzy topology on X.

PROPOSITION 3.12. Let X be a universal group and $(X,\mathcal{T})$ an fts. Then a fuzzy group G in X is an ftg iff the mapping f: $(x,y)\to xy^{-1}$ of $(G,\mathcal{T}_G)\times(G,\mathcal{T}_G)$ into $(G,\mathcal{T}_G)$ is RF-continuous.

PROOF. We can get from Proposition 2.15 that the mapping f is RF-continuous. Since the composition of RF-continuous mappings is RF-continuous, the composition $(x,y)\to(x,y^{-1})\to xy^{-1}$ is RF-continuous. Conversely, let e be the identity in X, then we have $G(x)\leq G(e)$ for all x in X from Proposition 3.8. Let i be a mapping of $(G,\mathcal{T}_G)\times(G,\mathcal{T}_G)$ such that $i:y\to(e,y)$. Then i is RF-continuous from Proposition 2.16 and $h:x\to x^{-1}$ is a composition of $x\to(e,x)\to ex^{-1}$. Hence h is RF-continuous. Similarly, $g:(x,y)\to(x,y^{-1})\to x(y^{-1})^{-1}$ is RF-continuous.

PROPOSITION 3.13. Let X be a topological group. Then X is an ftg iff for any Q-nbd W of $ab^{-1}(k)$, there are Q-nbds U of $a(k)$ and V of $b(k)$ such that $UV^{-1}\subset W$.

PROOF. It is similar to the proof in general topological groups and so is omitted.